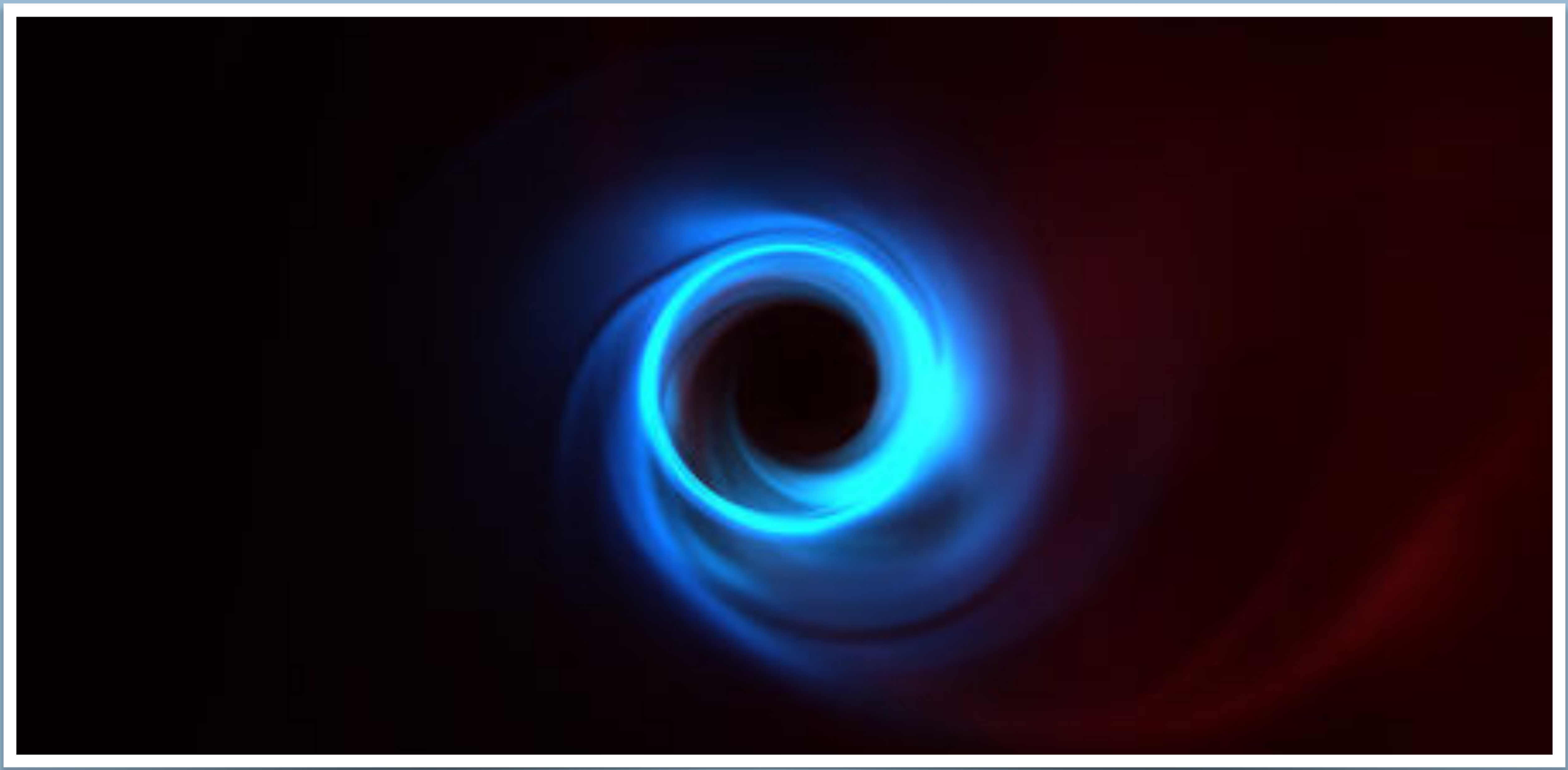

THE STABILITY OF BLACK HOLES

Elena Giorgi
Columbia University

OUTLINE

1. Motivation for the study of black holes
2. The mathematics of black holes
3. The stability problem for the Einstein equation
4. Black holes with matter fields

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Black holes are important astrophysical objects which are known to be overwhelmingly present in the universe.

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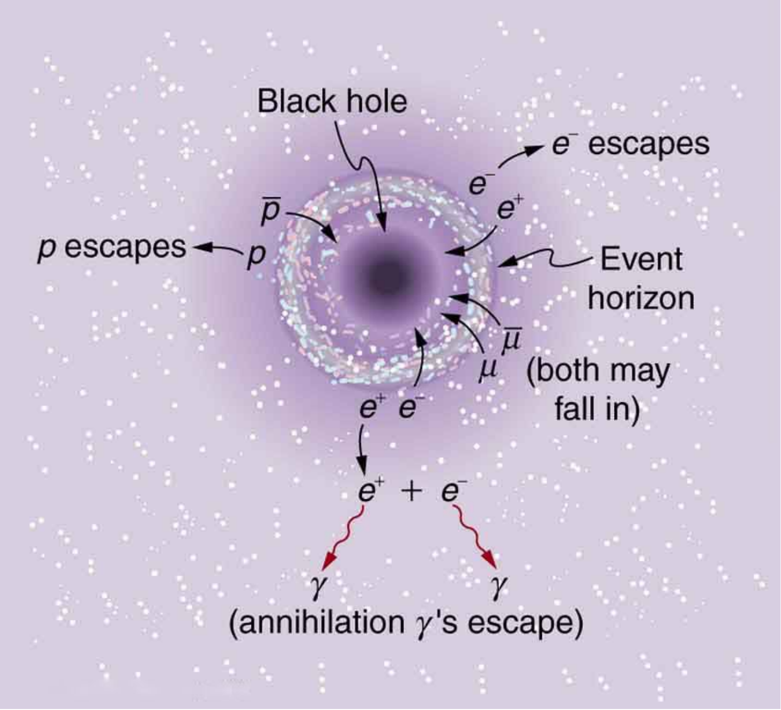
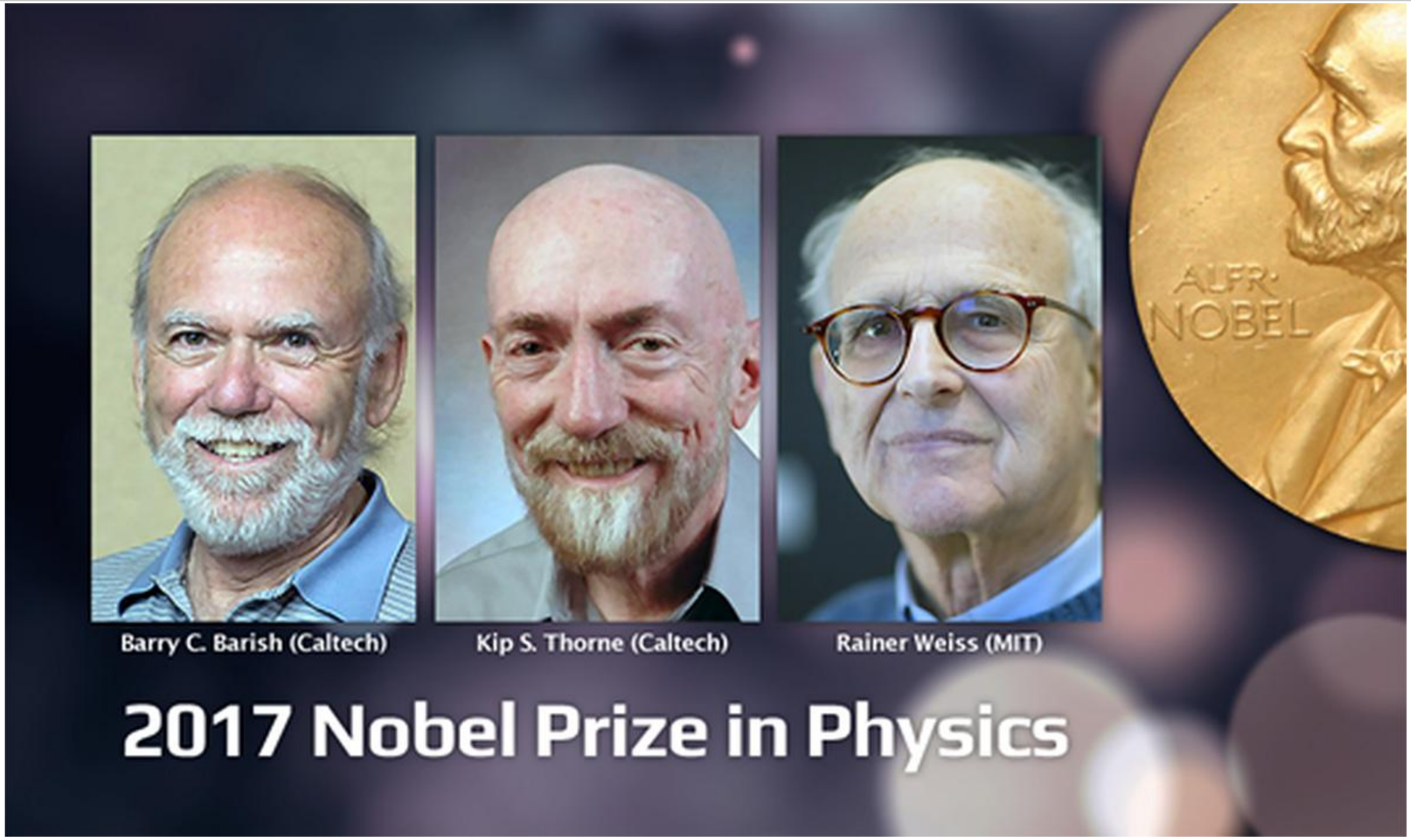


Because of their extreme nature in the realm of General Relativity, they are the perfect place to test the limit of the theory, and its unification with quantum mechanics for a quantum theory of gravity.

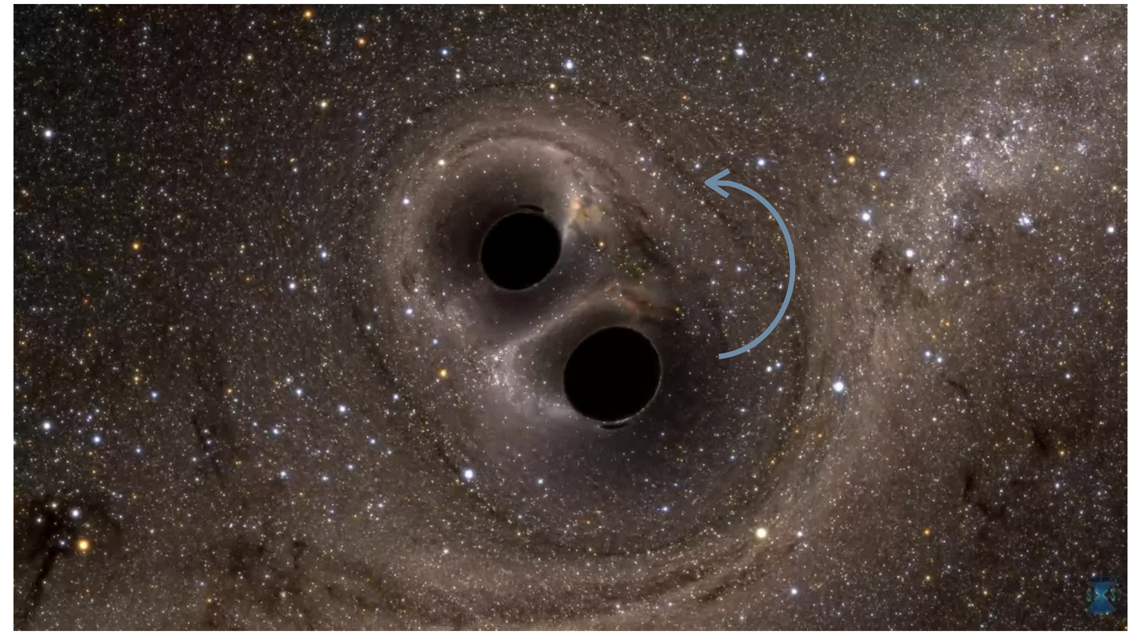


"for decisive contributions to the LIGO detector and the observation of gravitational waves"

Gravitational waves finally captured

On 14 September 2015, the universe's gravitational waves were observed for the very first time. The waves, which were predicted by Albert Einstein a hundred years ago, came from a collision between two black holes. It took 1.3 billion years for the waves to arrive at the LIGO detector in the USA.

LIGO observed the signal emitted by the merger of two black holes rotating around each other



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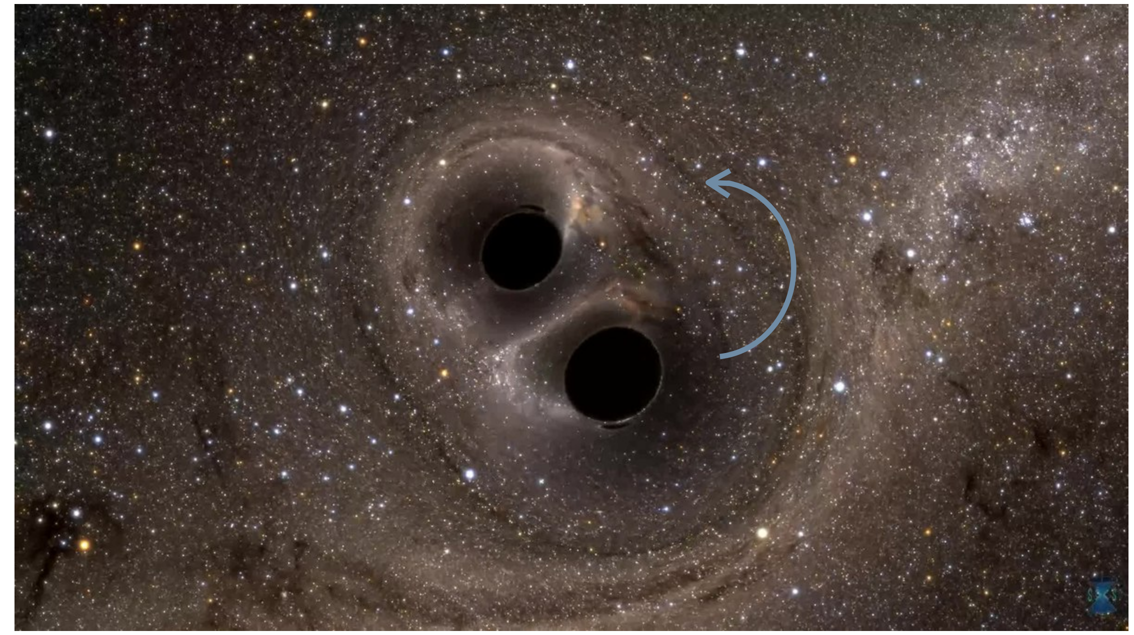

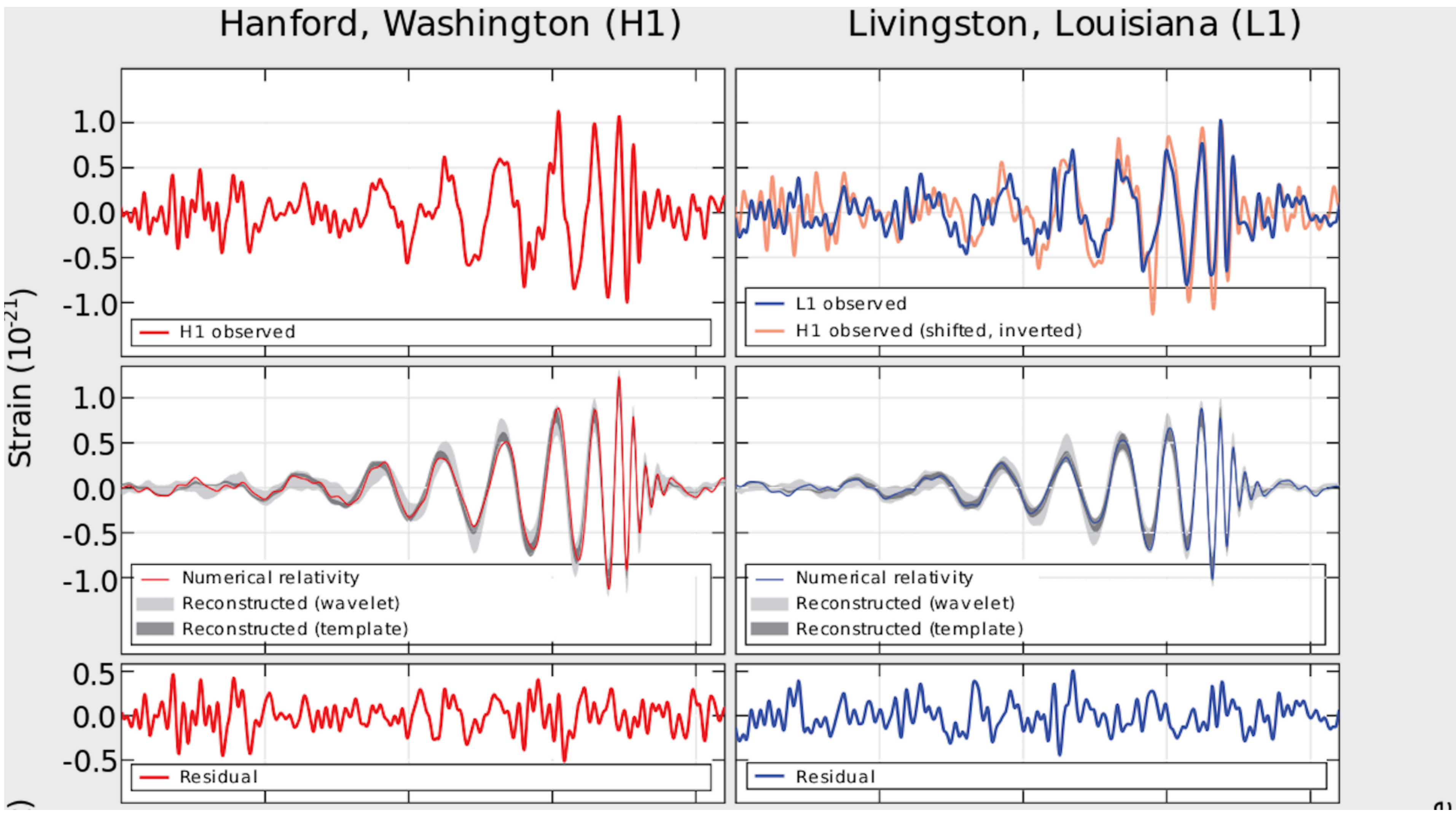

14th September 2015 GW observation

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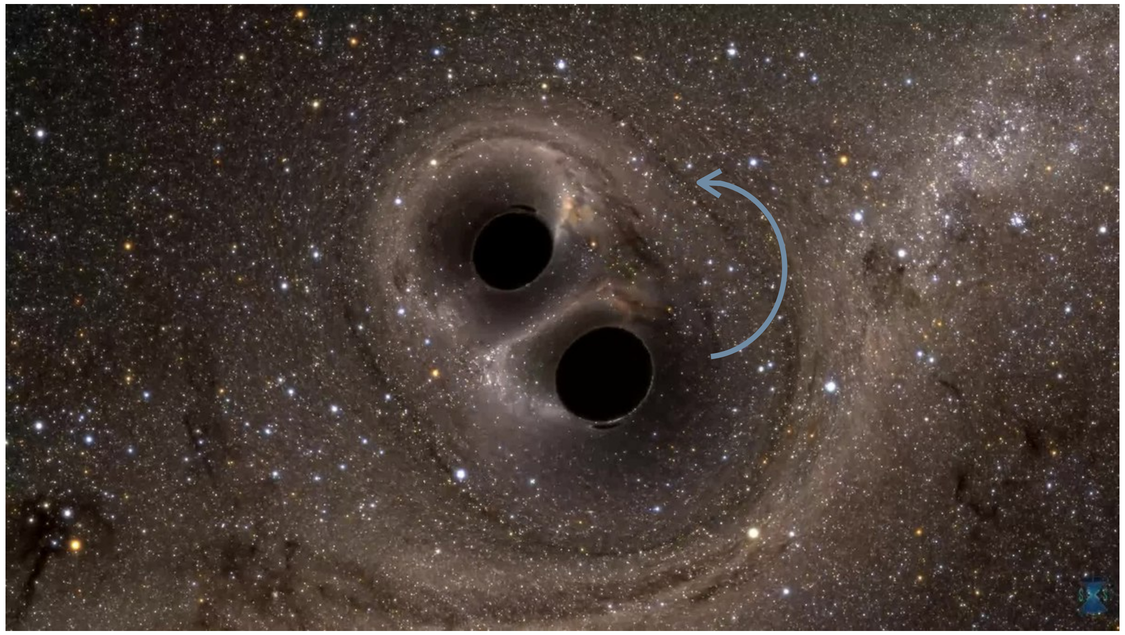
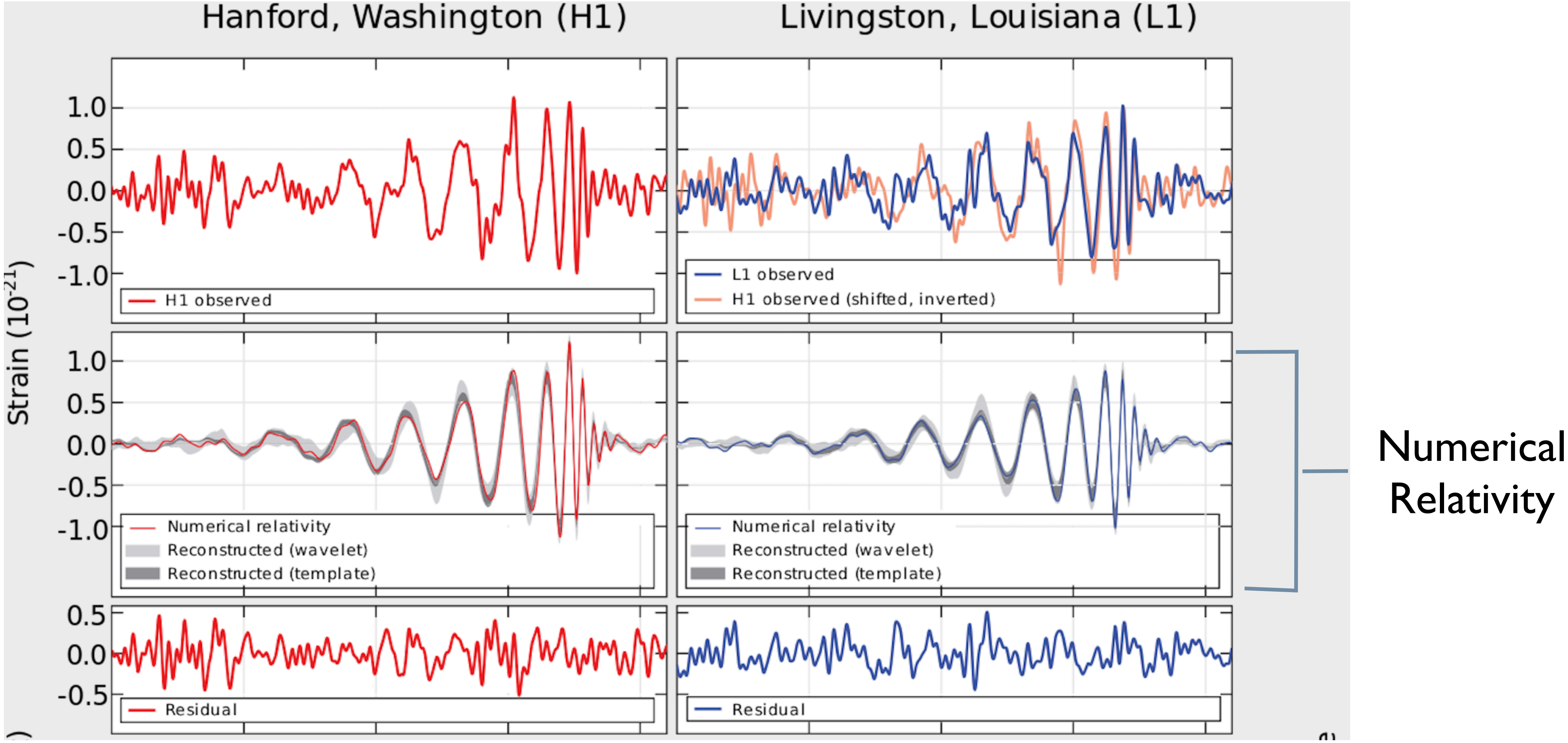
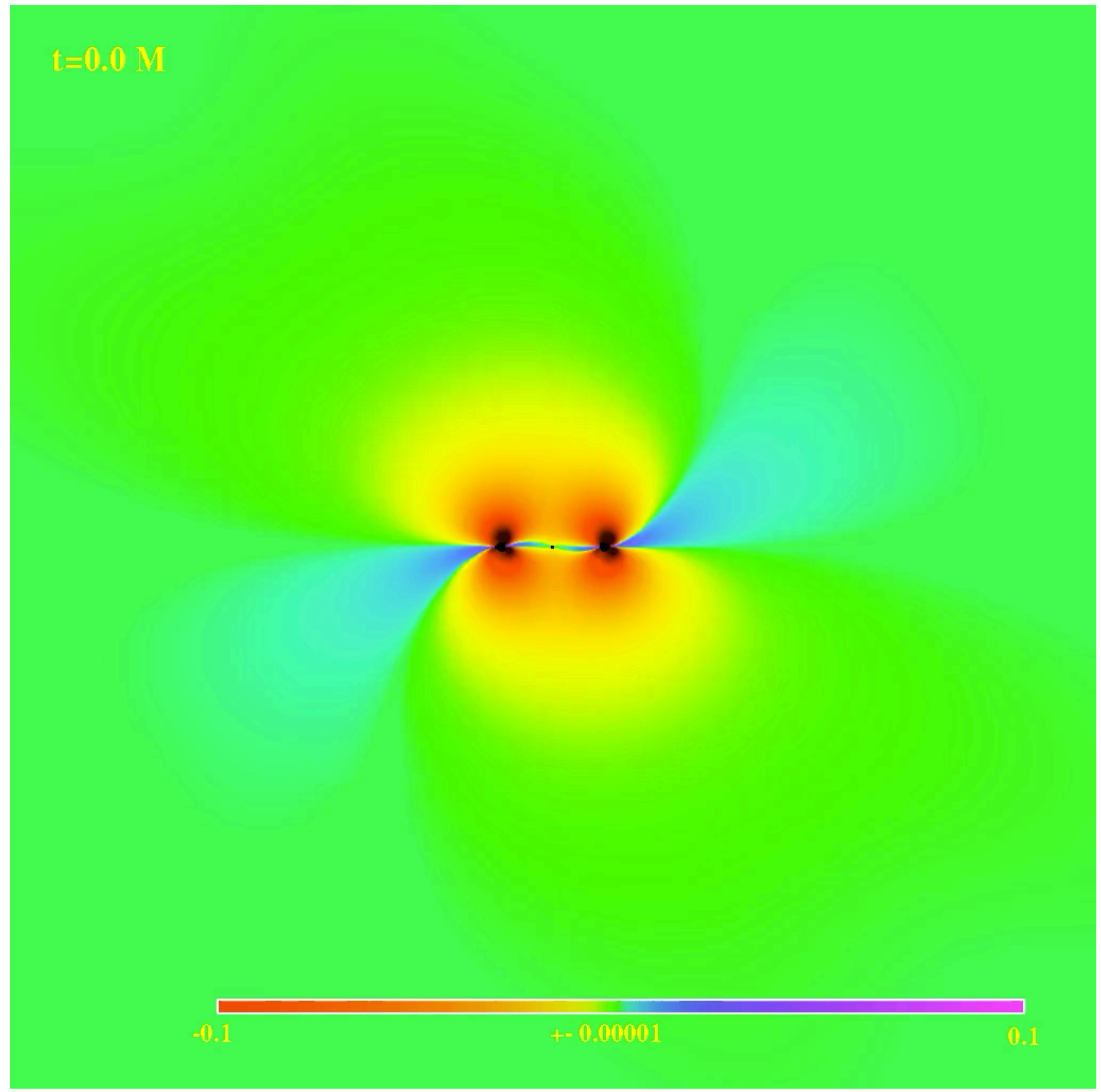




It is remarkable that the first code to simulate the merger of two black holes was only obtained in 2005 by Frans Pretorius, just ten years before the LIGO first observation.



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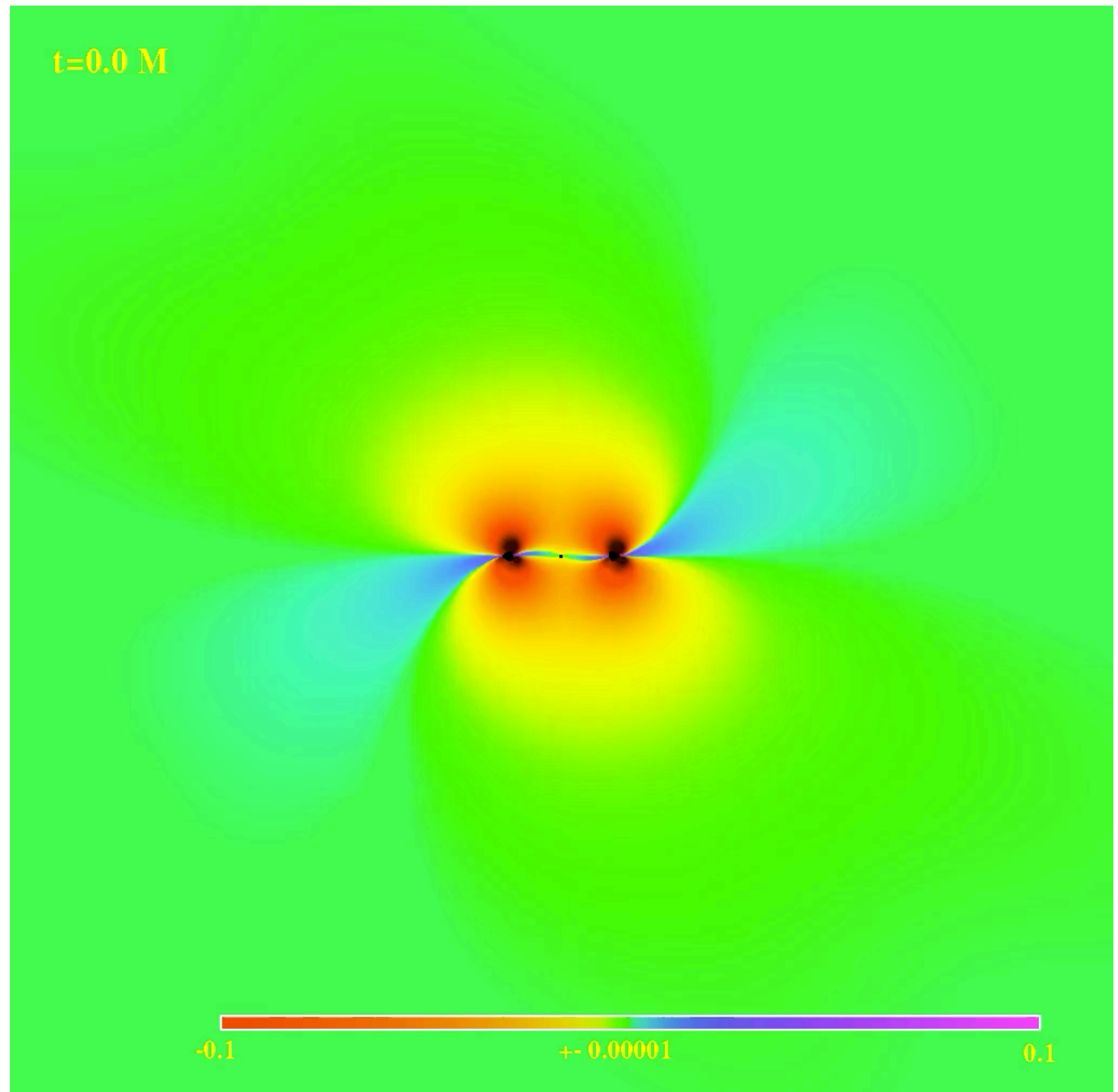
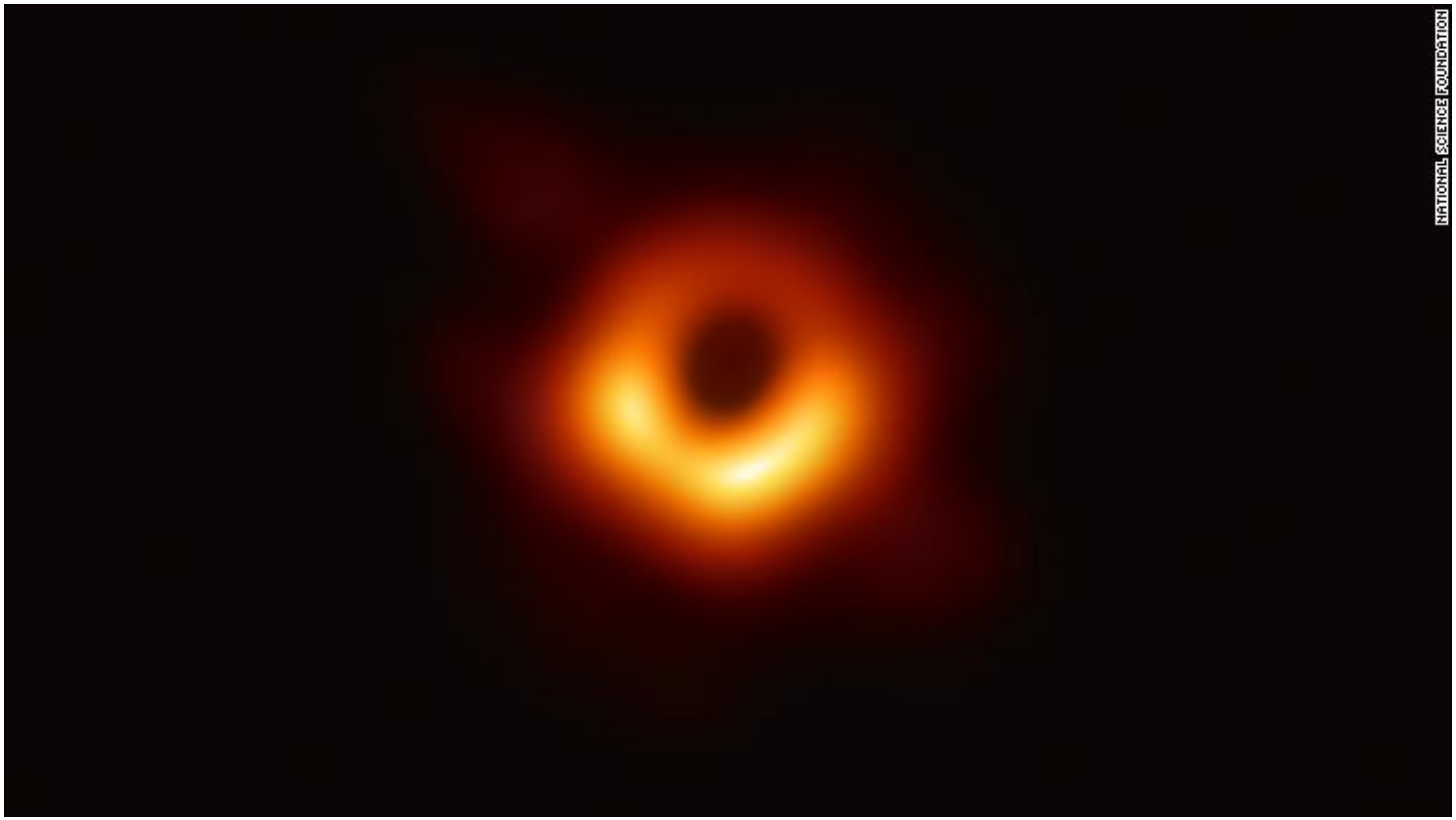


Frans Pretorius' simulation, http://physics.princeton.edu/~fpretori/

Another remarkable evidence for the existence of black holes was given in April 2019 by the first image of M87 obtained by the Event Horizon Telescope



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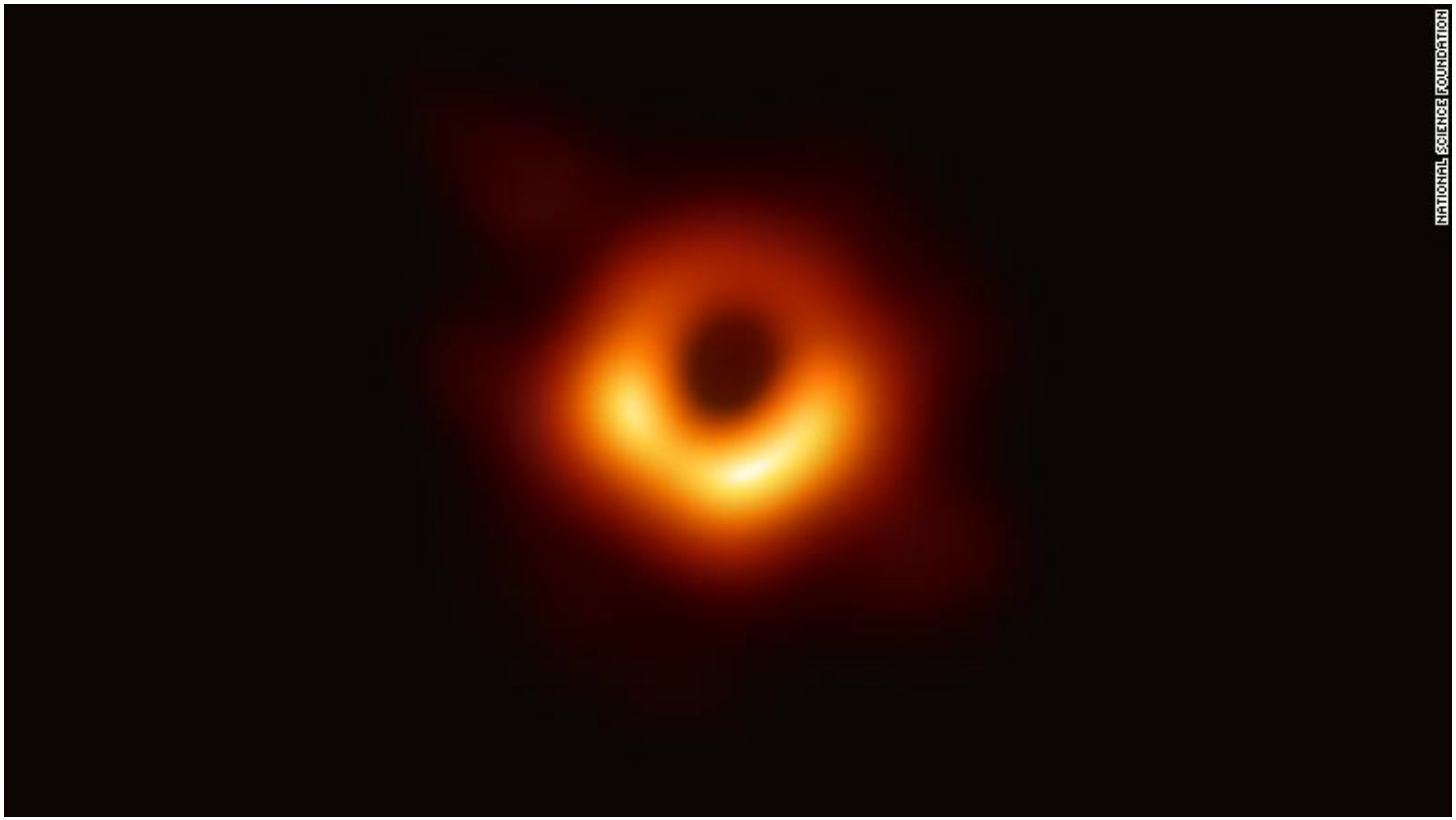


2020 Breakthrough Prize in Fundamental Physics

- **The Event Horizon Telescope Collaboration**

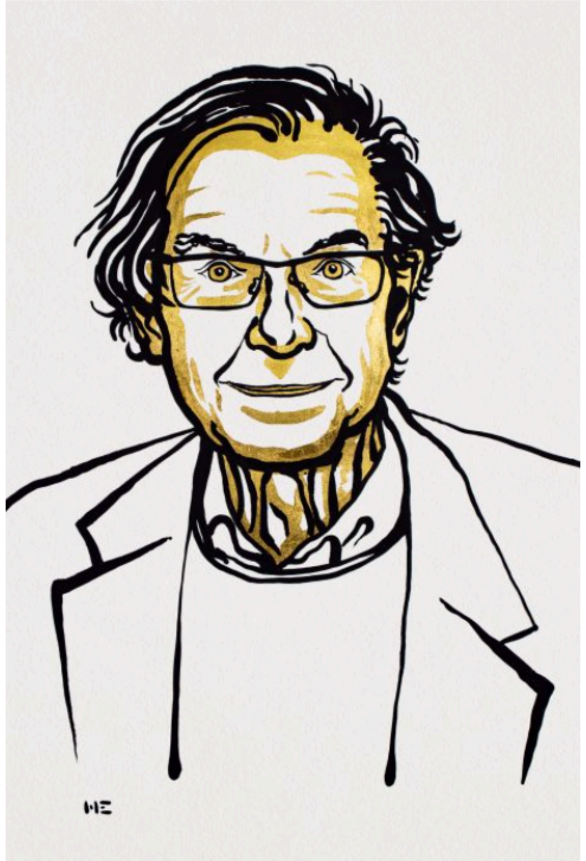
Citation: For the first image of a supermassive black hole, taken by means of an Earth-sized alliance of telescopes.

Event Horizon Telescope

The Black Hole Shadow in M 87

Cover Pages, 2019 April 11

The Nobel Prize in Physics 2020



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Roger Penrose

Prize share: 1/2

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Reinhard Genzel

Prize share: 1/4

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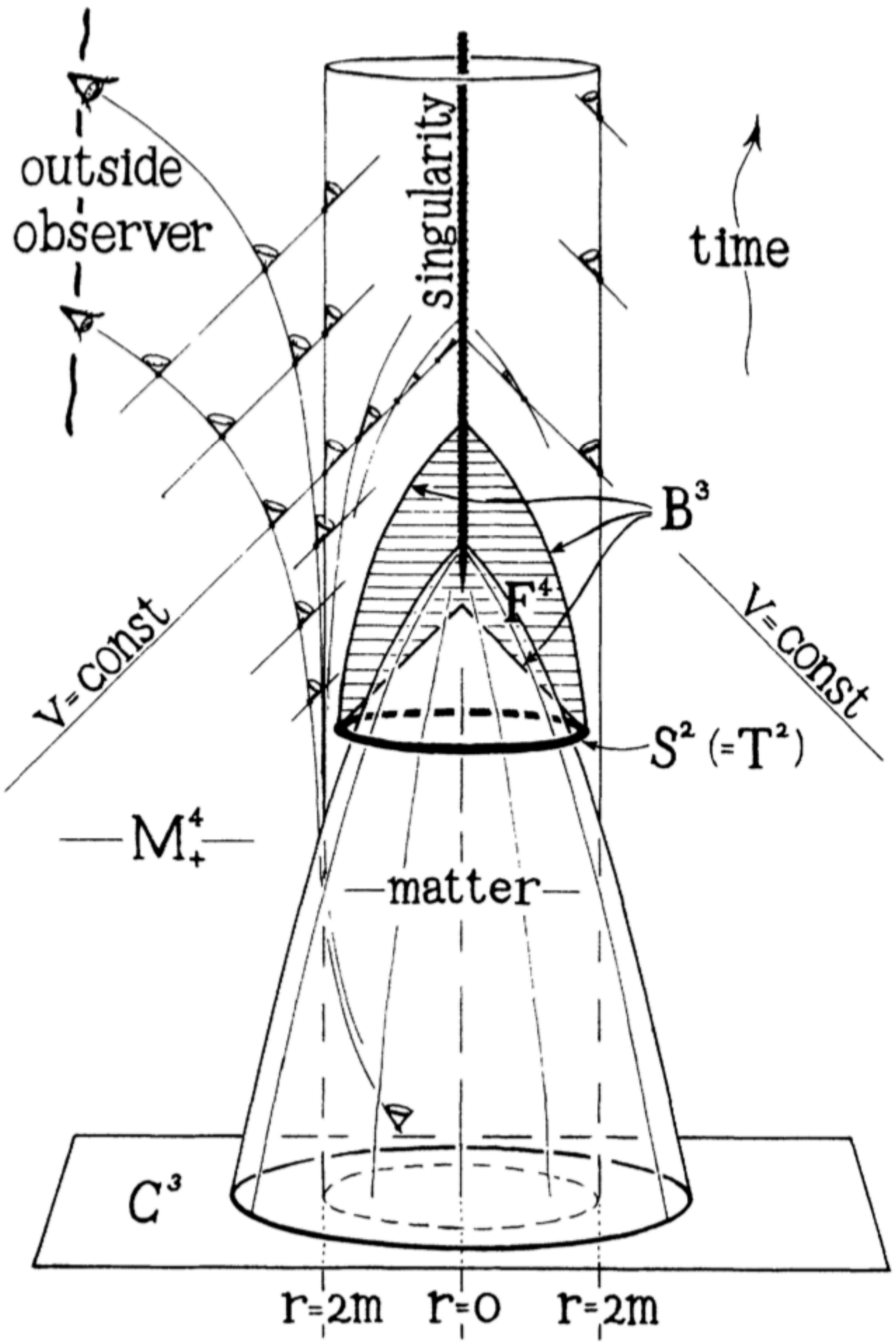
Andrea Ghez

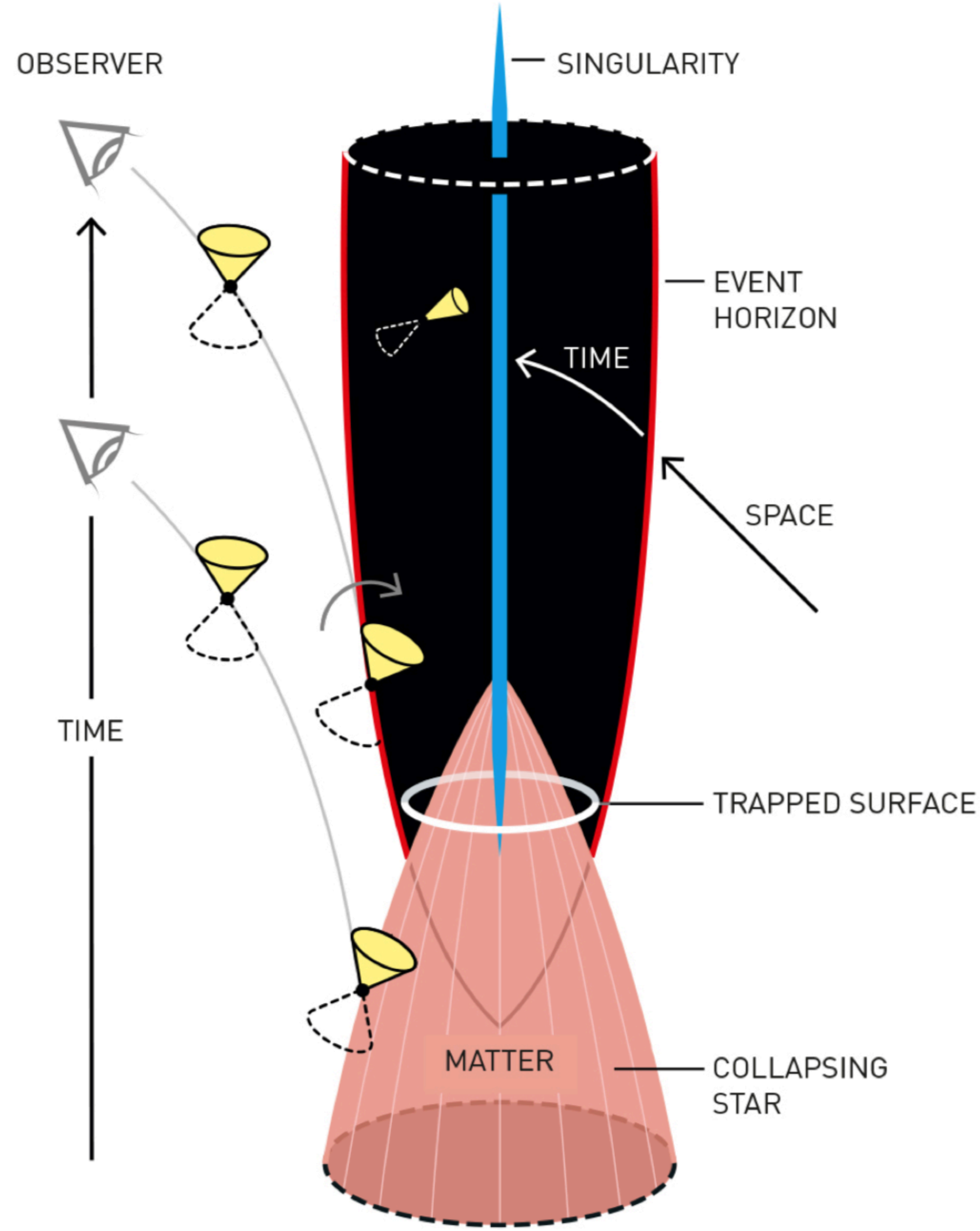
Prize share: 1/4

"for the discovery that black hole formation is a robust prediction of the general theory of relativity"

"for the discovery of a supermassive compact object at the centre of our galaxy"



Penrose, Phys. Rev. Lett. 14, 57 (1965)



Scientific background on the Nobel Prize in Physics 2020, https://www.nobelprize.org

How can mathematics help?

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NEWTON VS EINSTEIN

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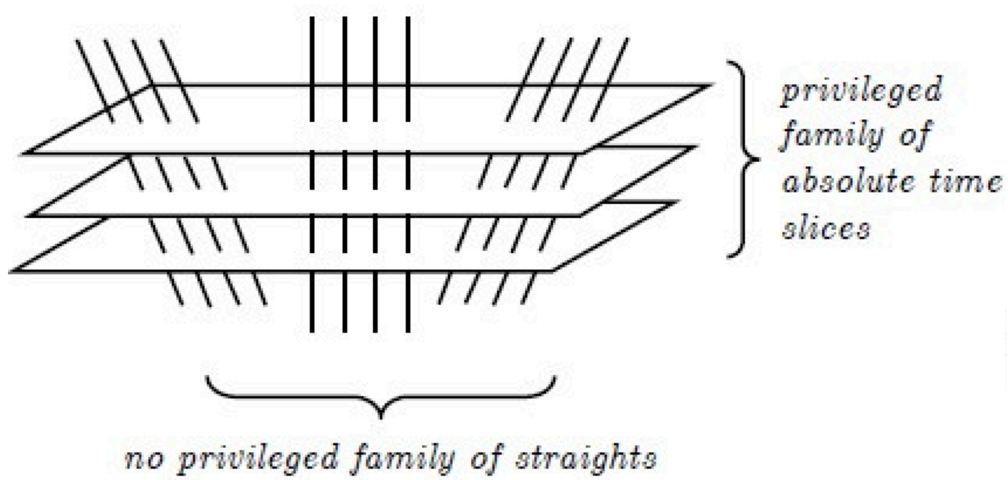


Newtonian spacetime:

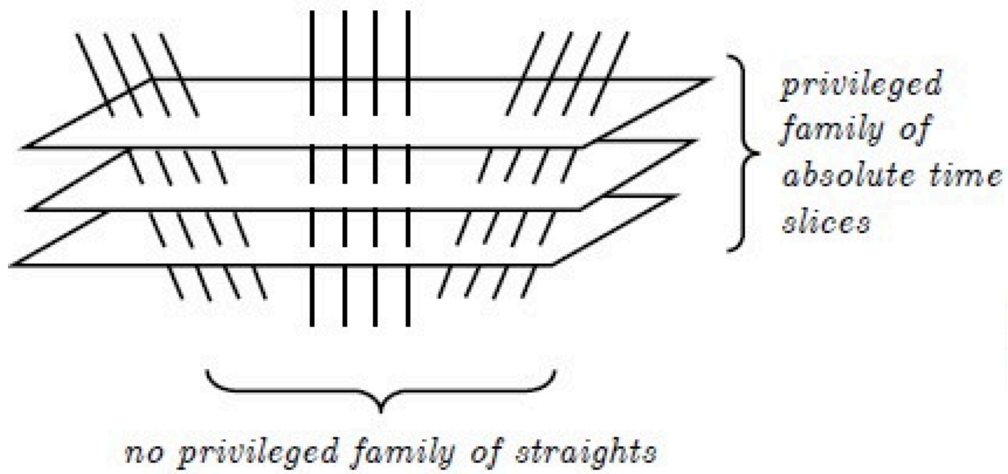
3 dimensional flat space

1 dimensional absolute time

The spatial separation is conserved:

$$ds = \sqrt{dx^2 + dy^2 + dz^2}$$

NEWTON VS EINSTEIN



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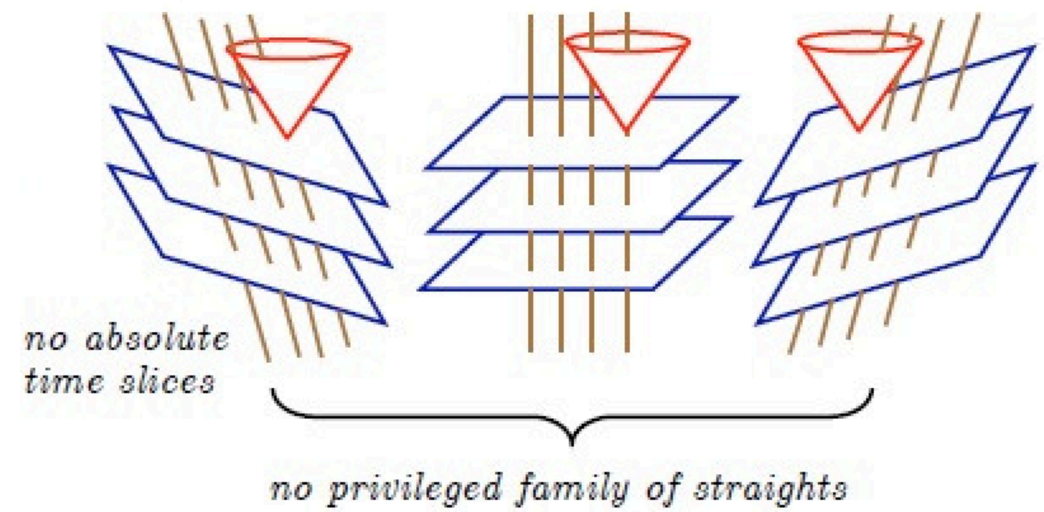
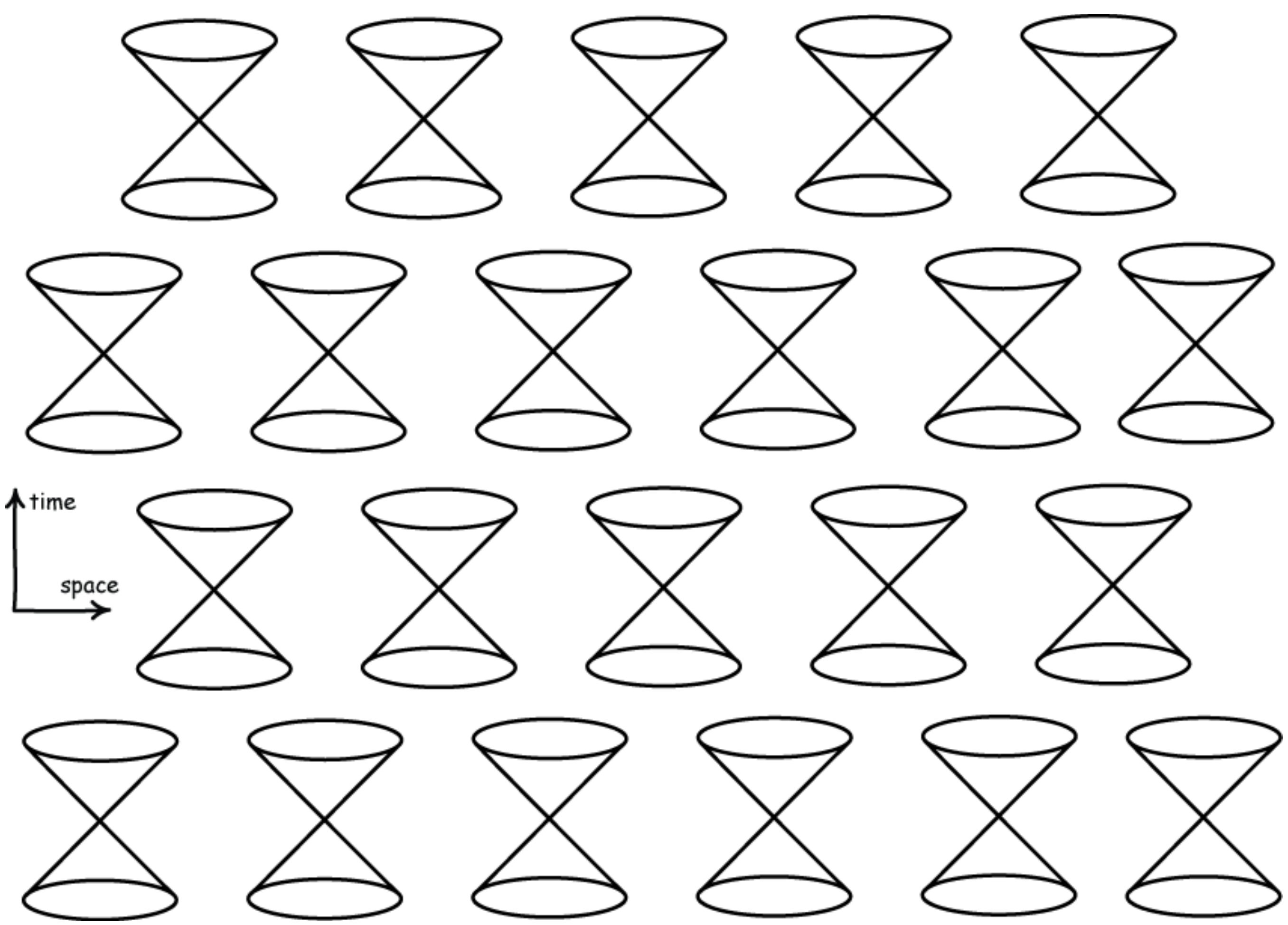


Minkowski spacetime:
4 dimensional flat spacetime

The spacetime separation is conserved:

$$ds = \sqrt{-c^2 dt^2 + dx^2 + dy^2 + dz^2}$$

THE SPACETIME OF SPECIAL RELATIVITY (1905)



The Minkowski spacetime is the $3+1$ flat metric on $\mathbb{R}^{3+1}$

$$g_m = -\,dt^2 + dx^2 + dy^2 + dz^2$$

which is the Lorentzian equivalent of the Euclidean space in Riemannian geometry

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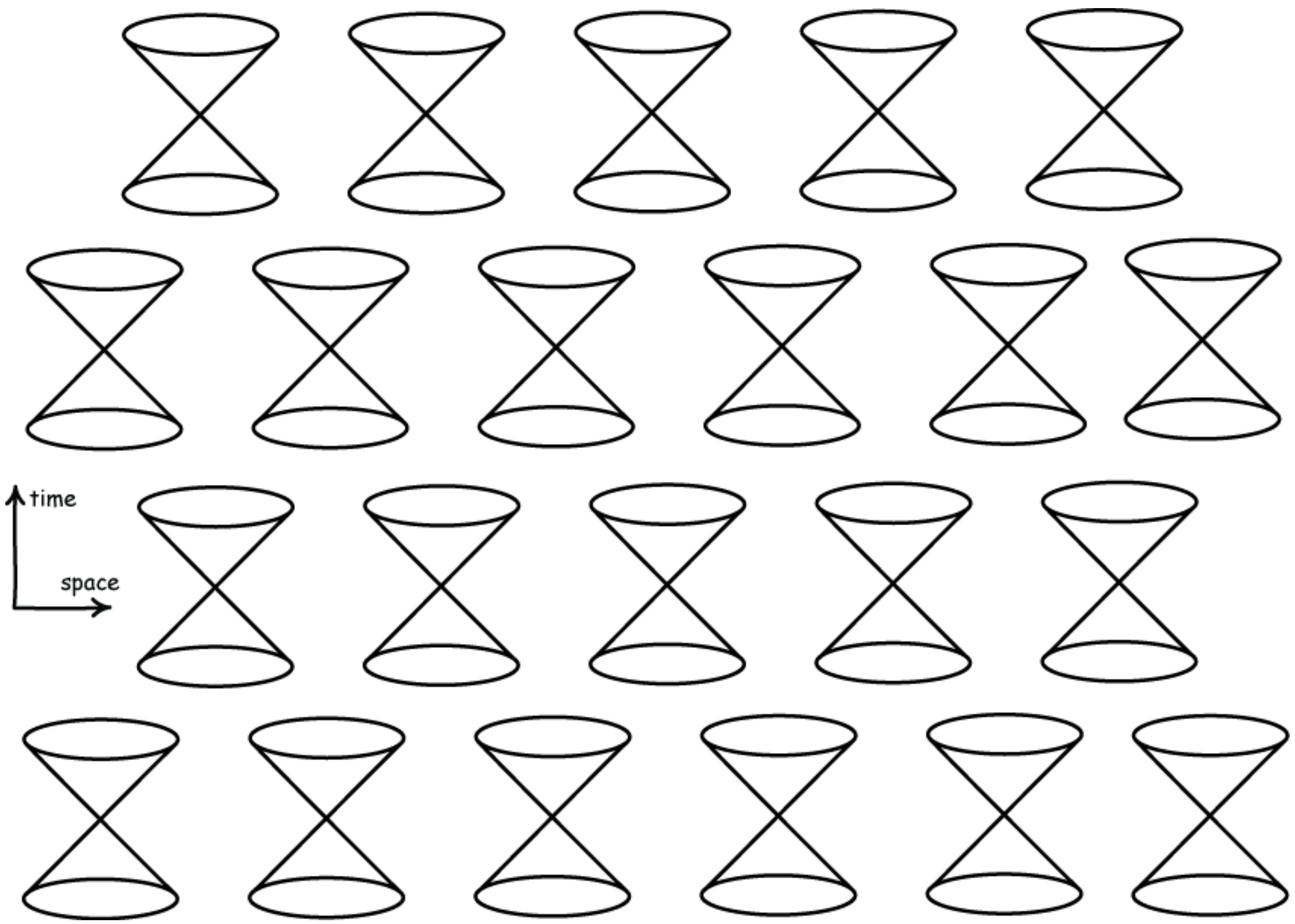

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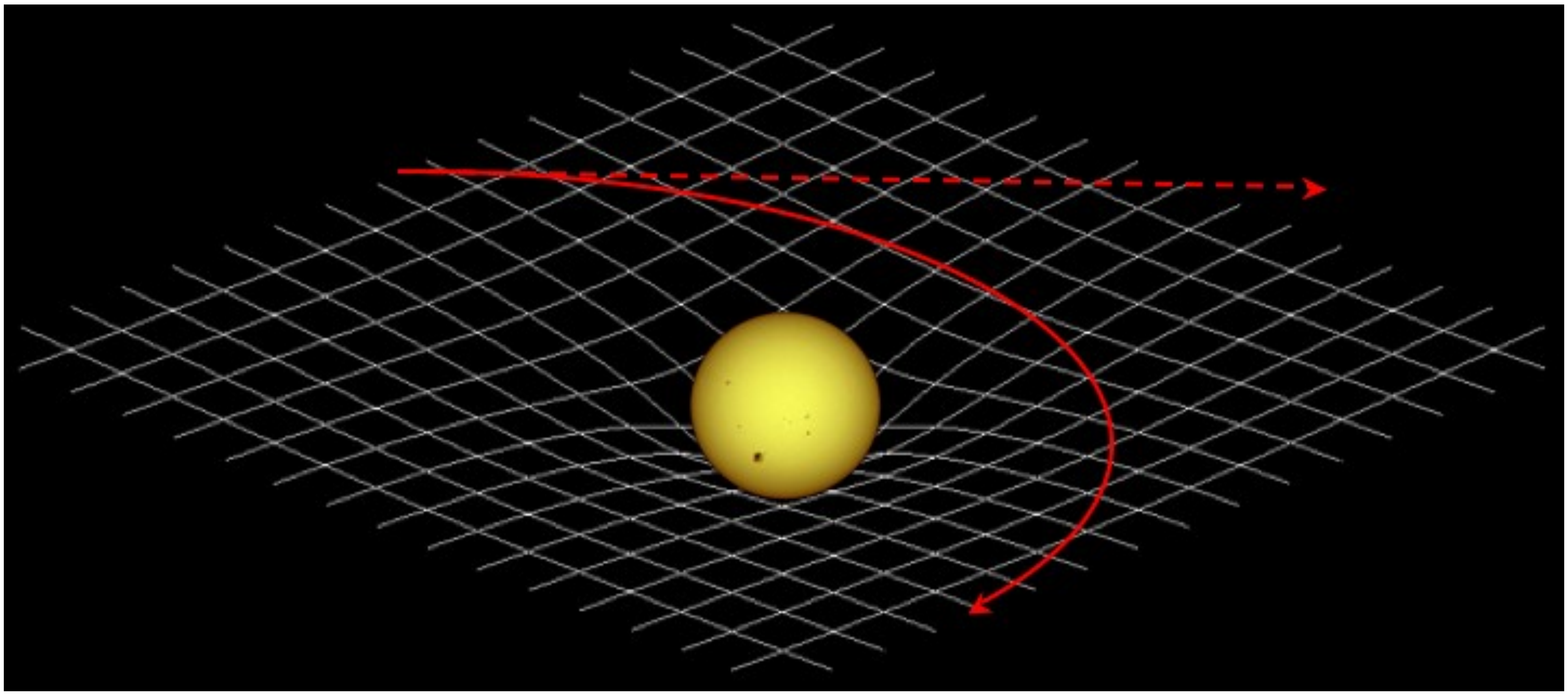
What happens in the presence of a massive object?

THE SPACETIME OF A STAR (1915)



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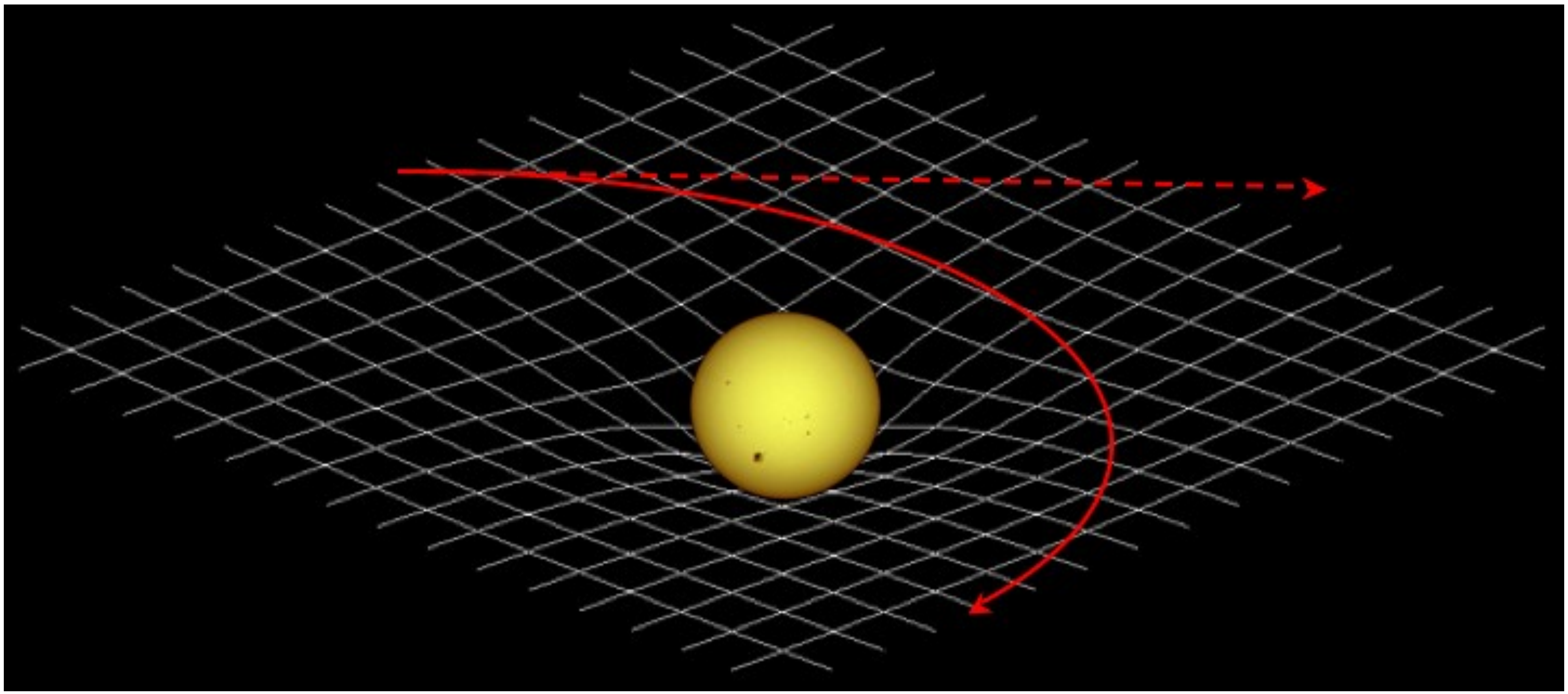
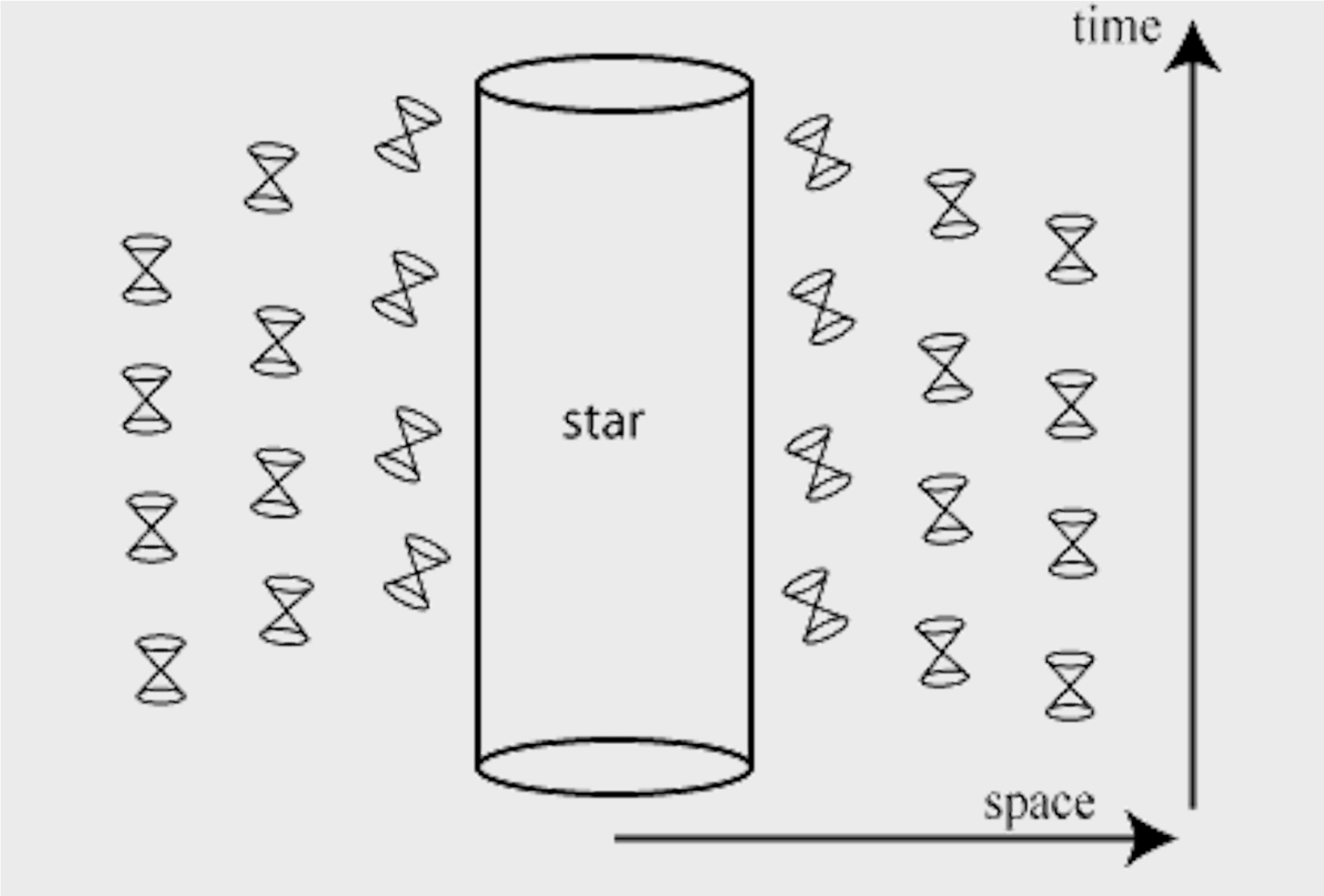
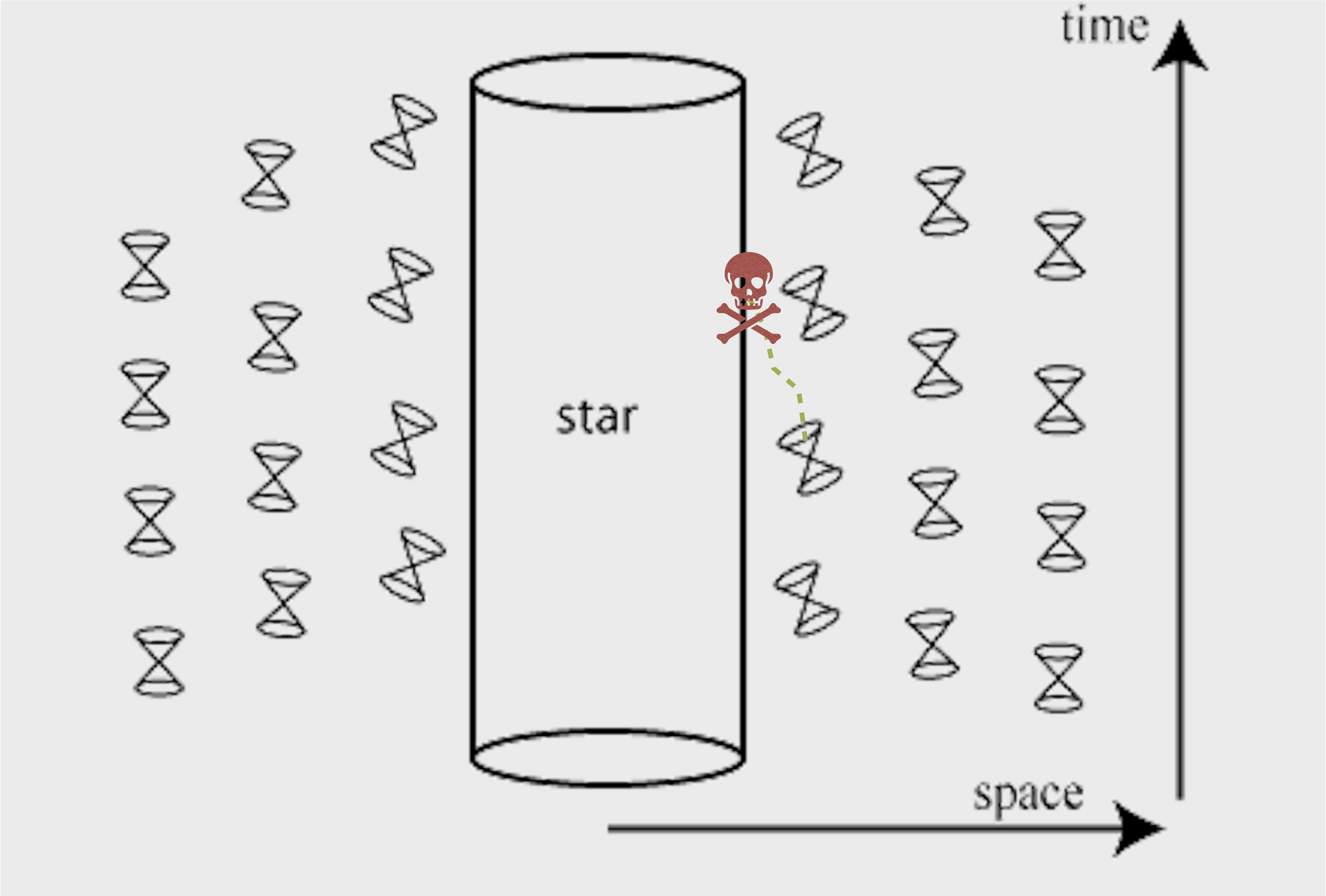


"Spacetime tells matter how to move;
matter tells spacetime how to curve"
John Wheeler


time
star
space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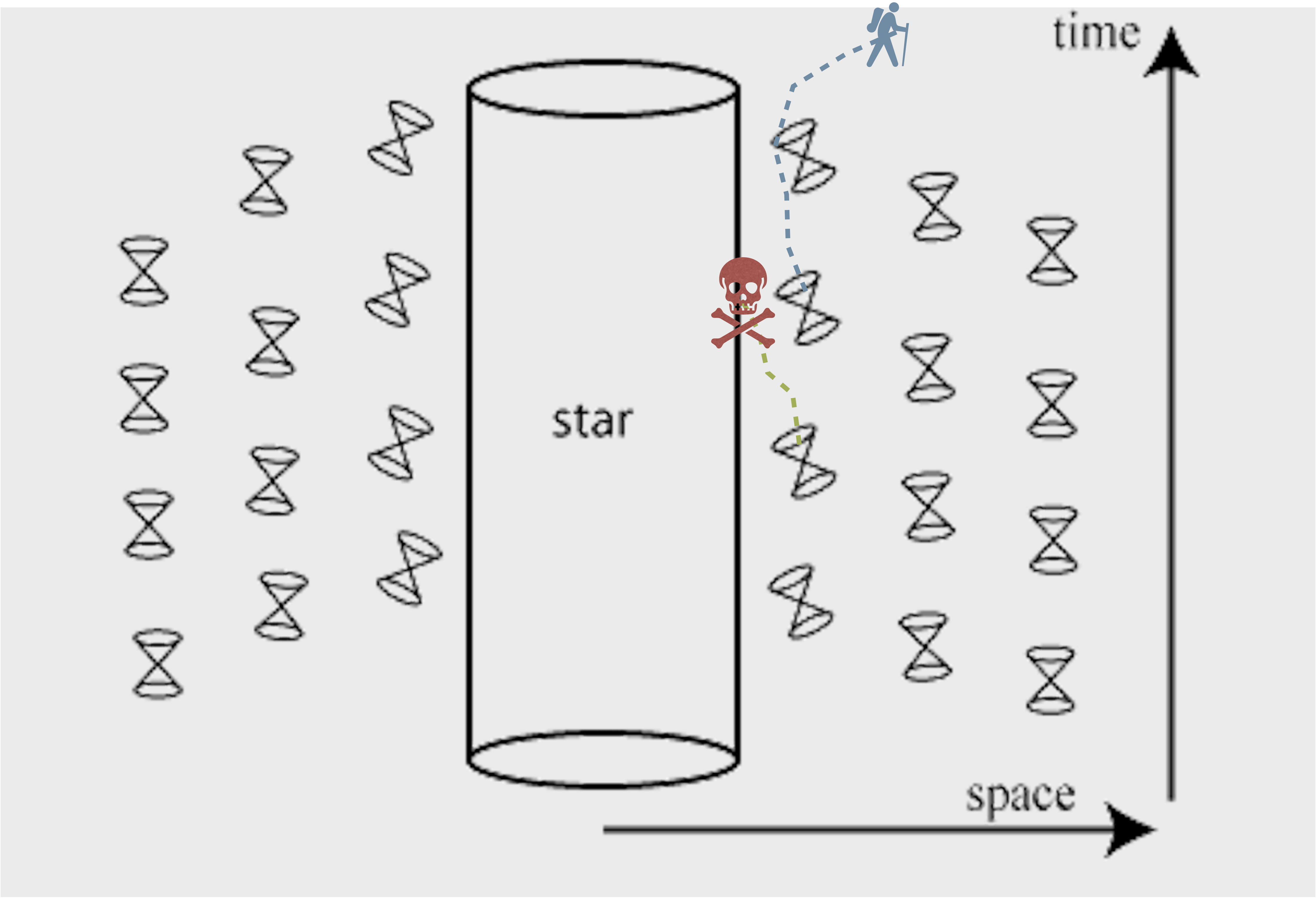
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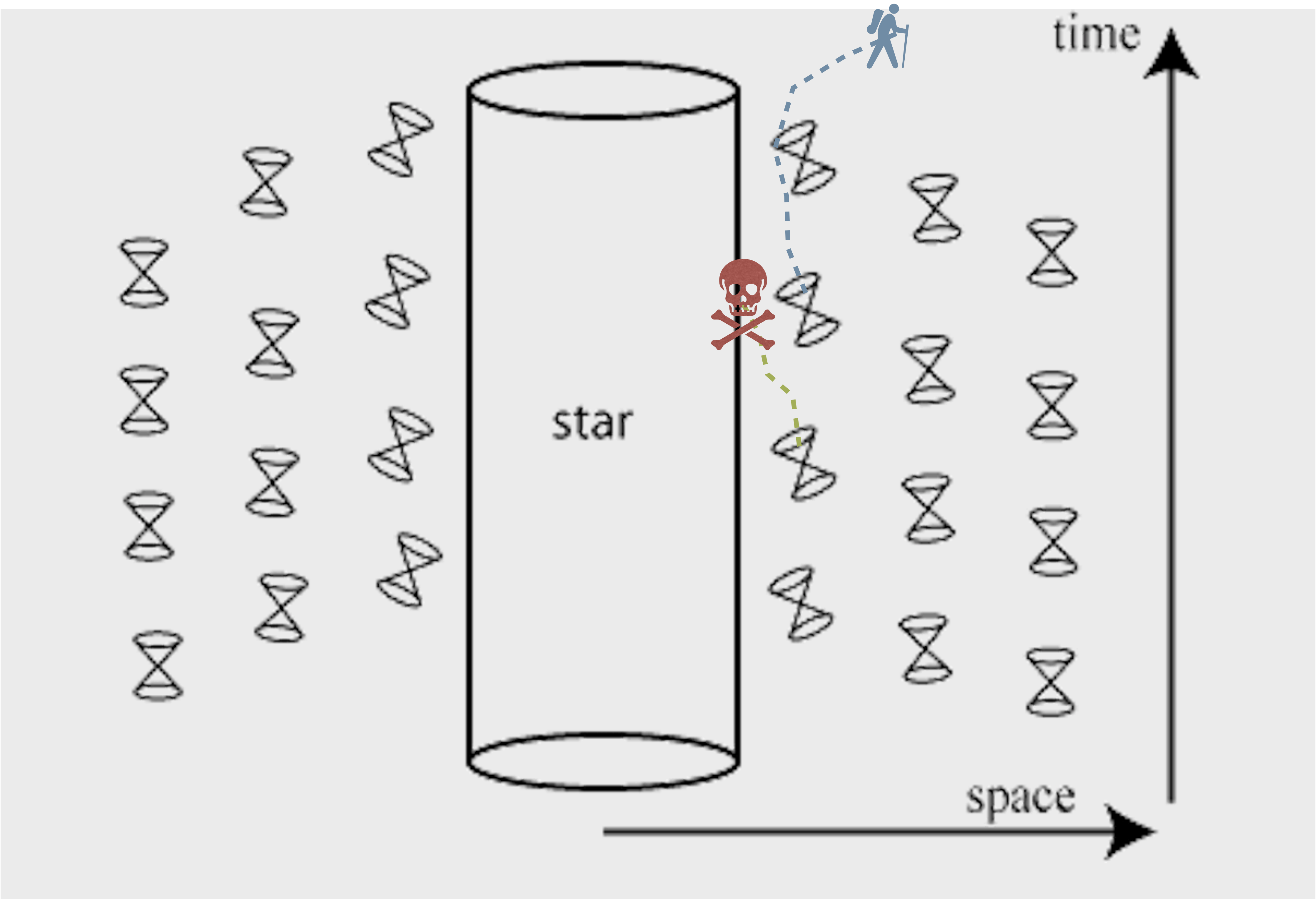
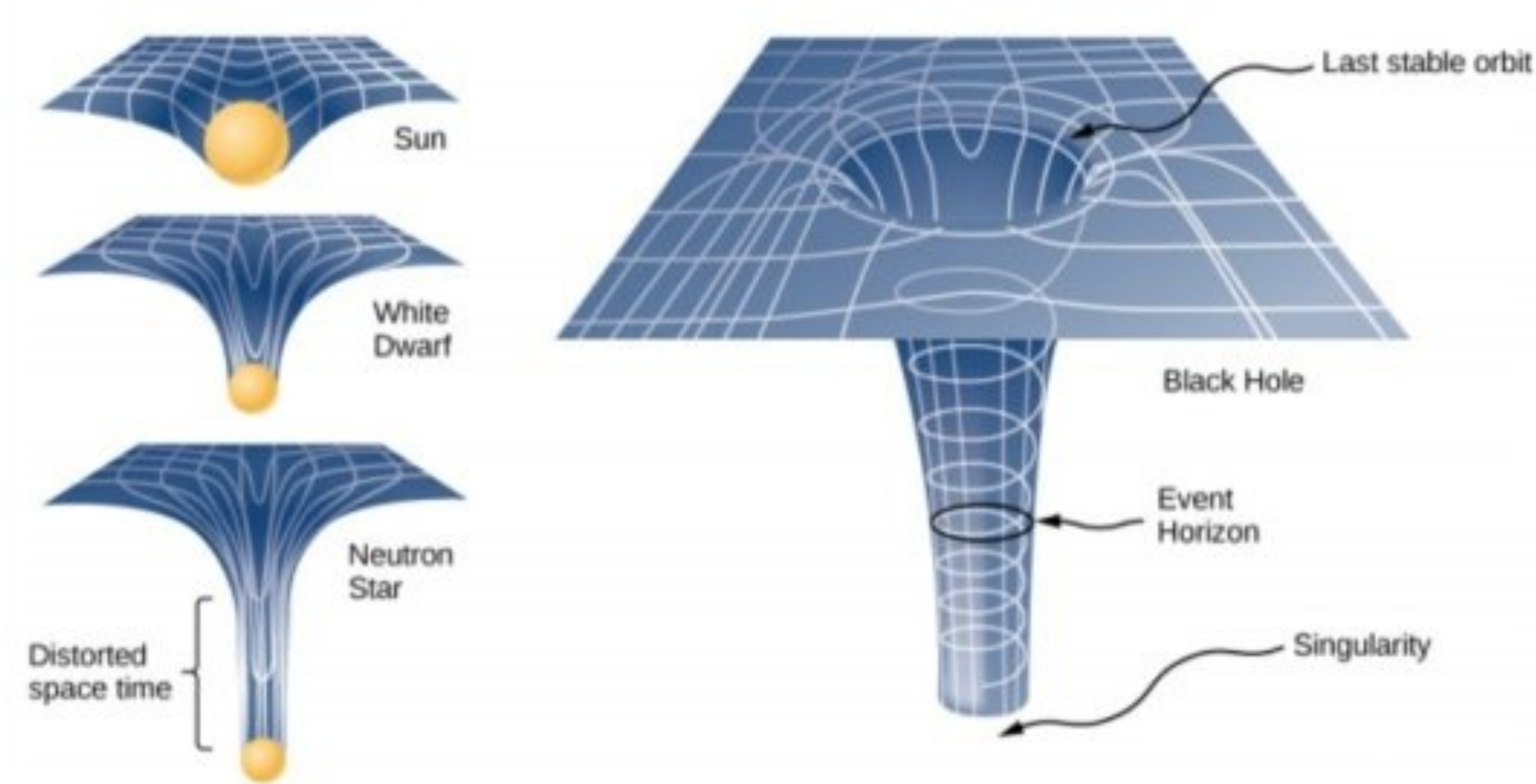
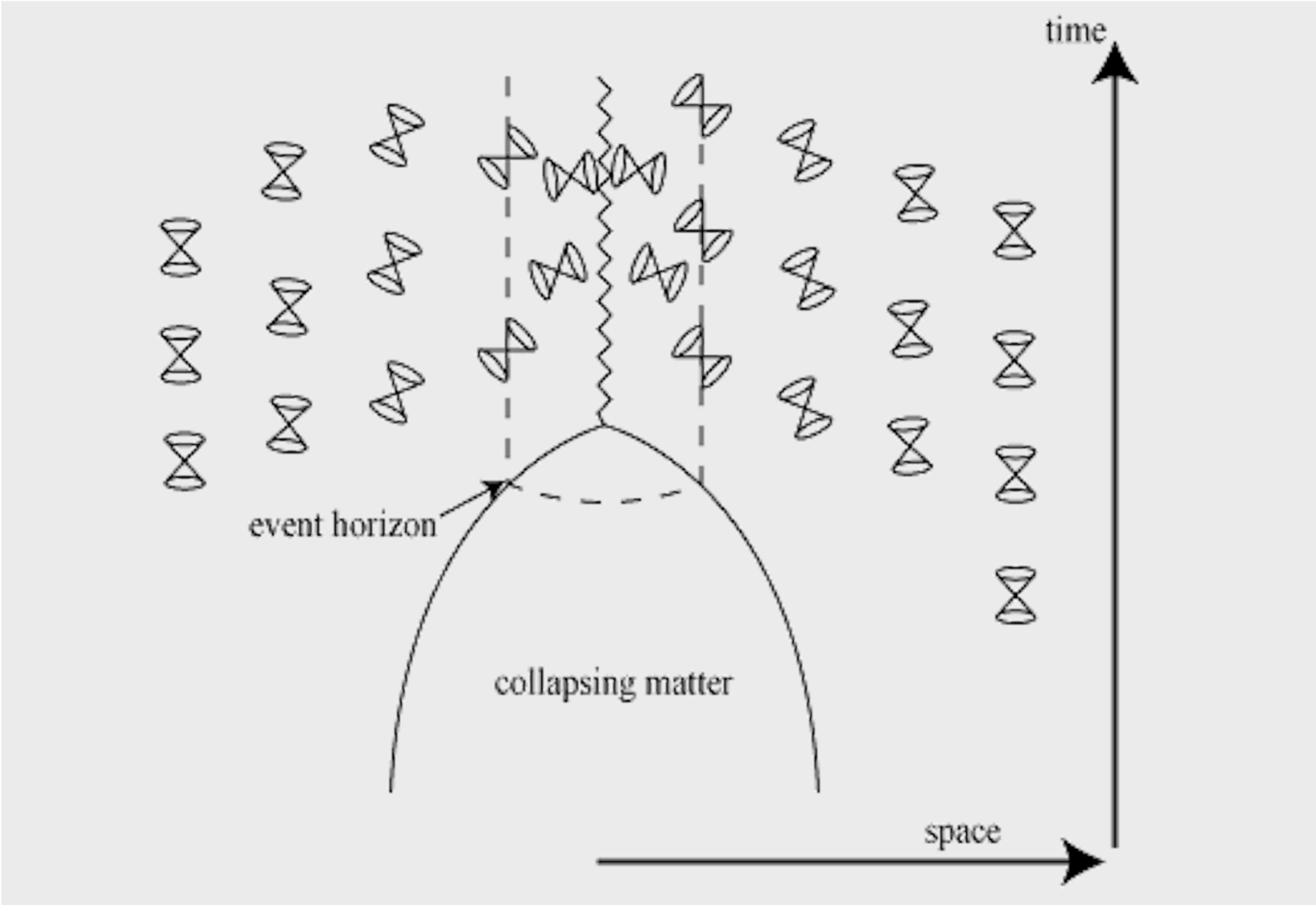
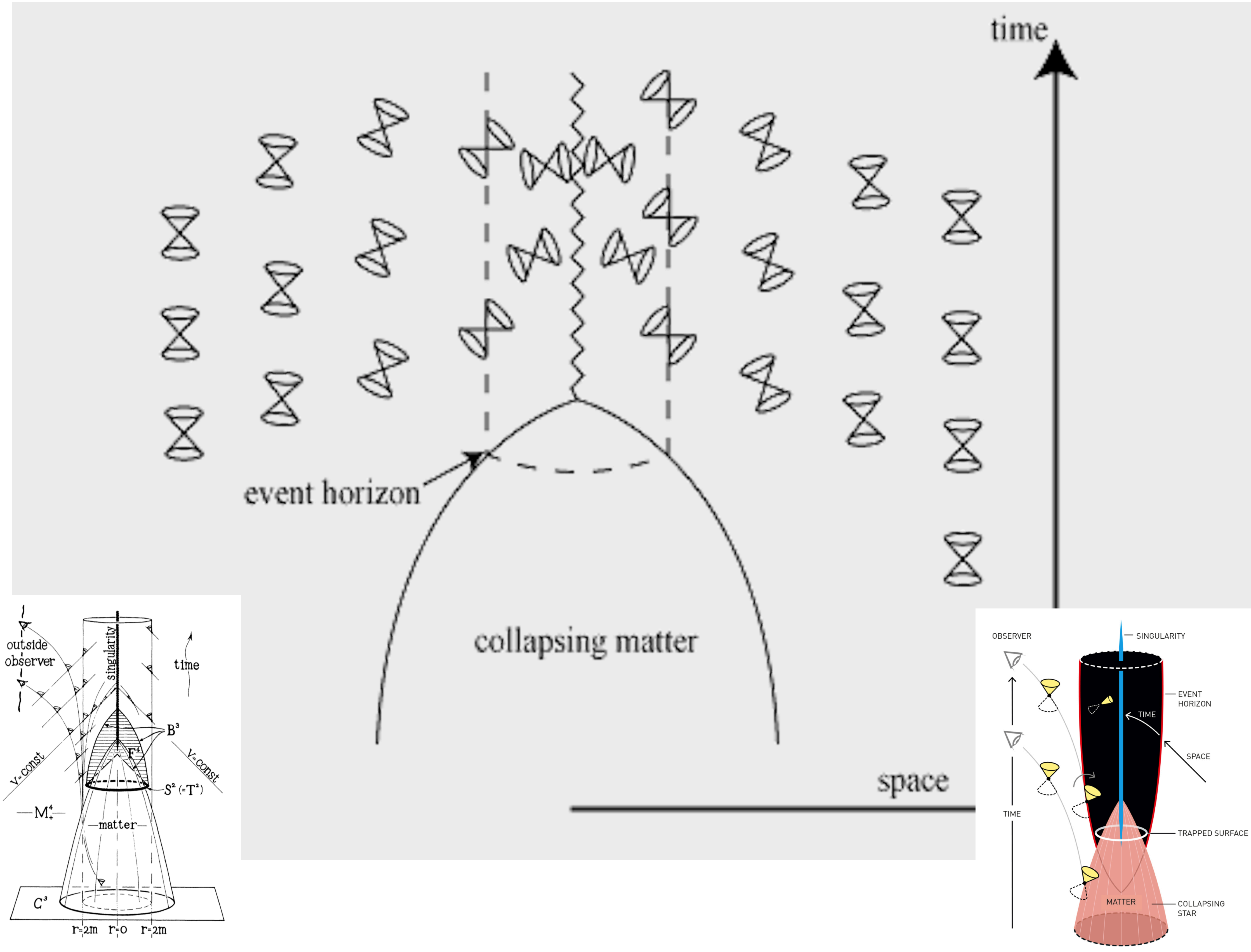


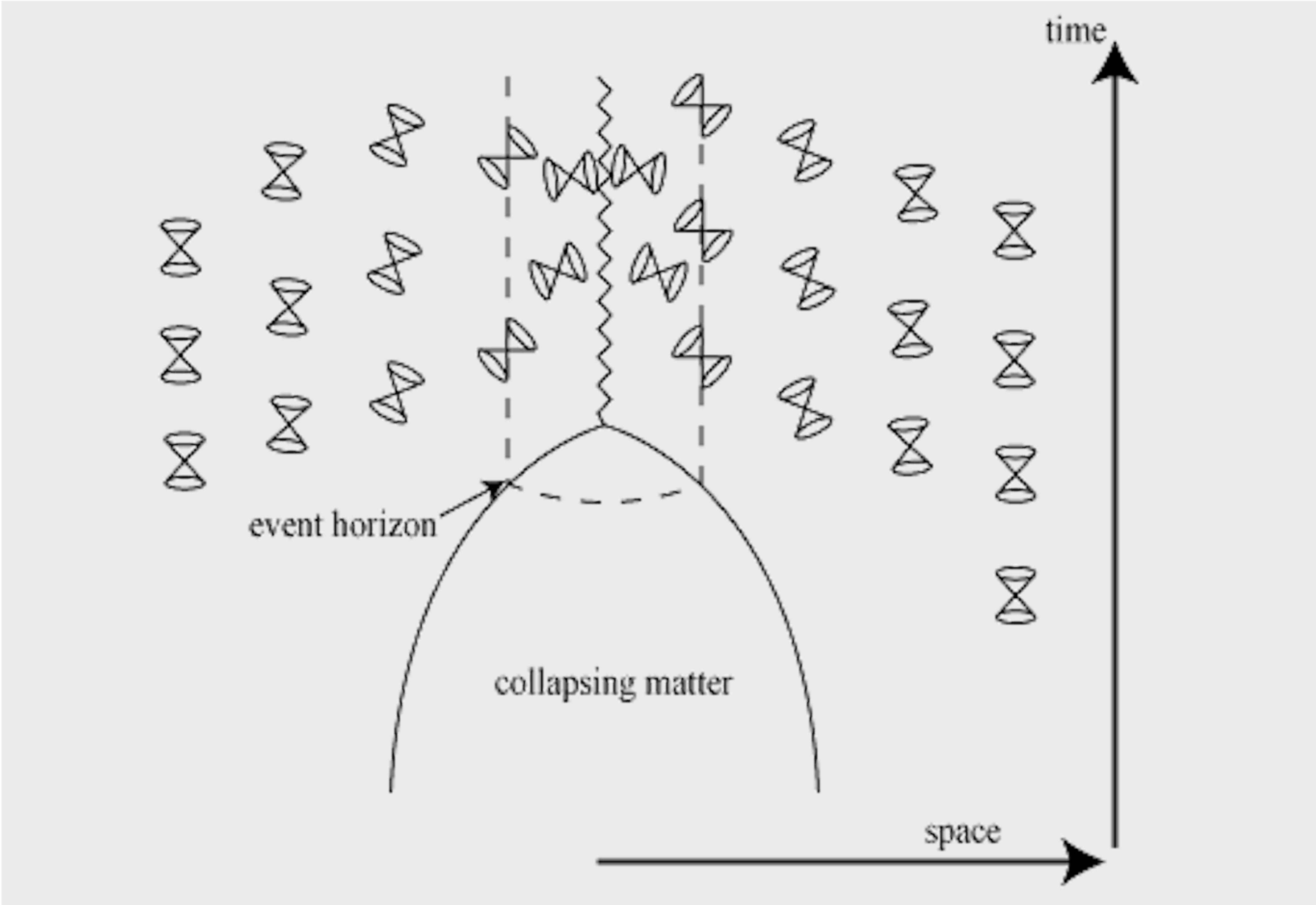
The geometry radically changes if the star becomes more and more massive and dense



The spacetime gets distorted:
the overall geometry of the light cones changes,
and a region where not even light can escape forms.


time
event horizon
collapsing matter
space


time
event horizon
collapsing matter
space
outside observer
singularity
time
B³
F⁴
S² (=T²)
v=const
v=const
M⁴₊
matter
C³
r=2m r=0 r=2m
OBSERVER
SINGULARITY
EVENT HORIZON
TIME
SPACE
TIME
TRAPPED SURFACE
MATTER
COLLAPSING STAR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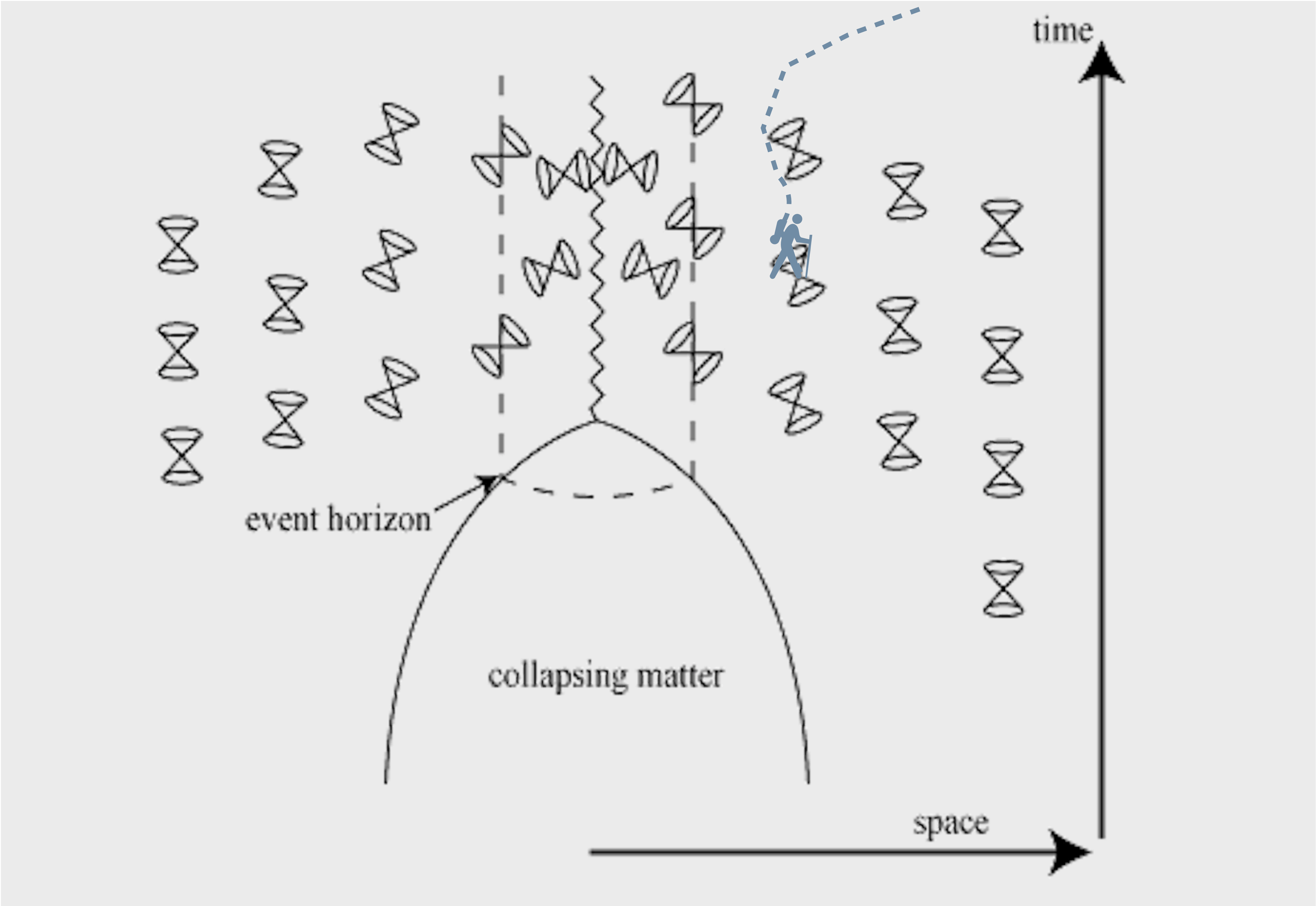
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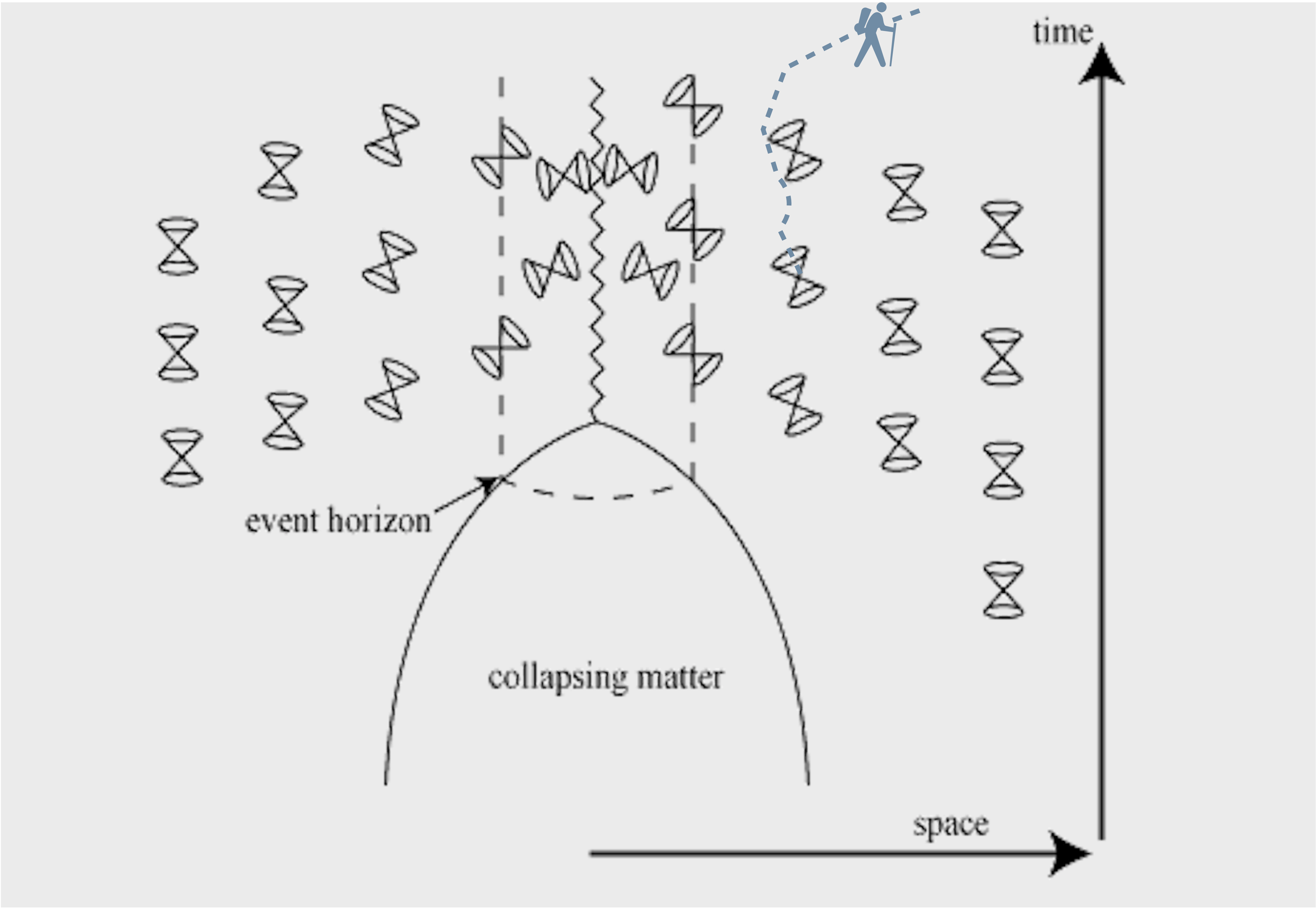
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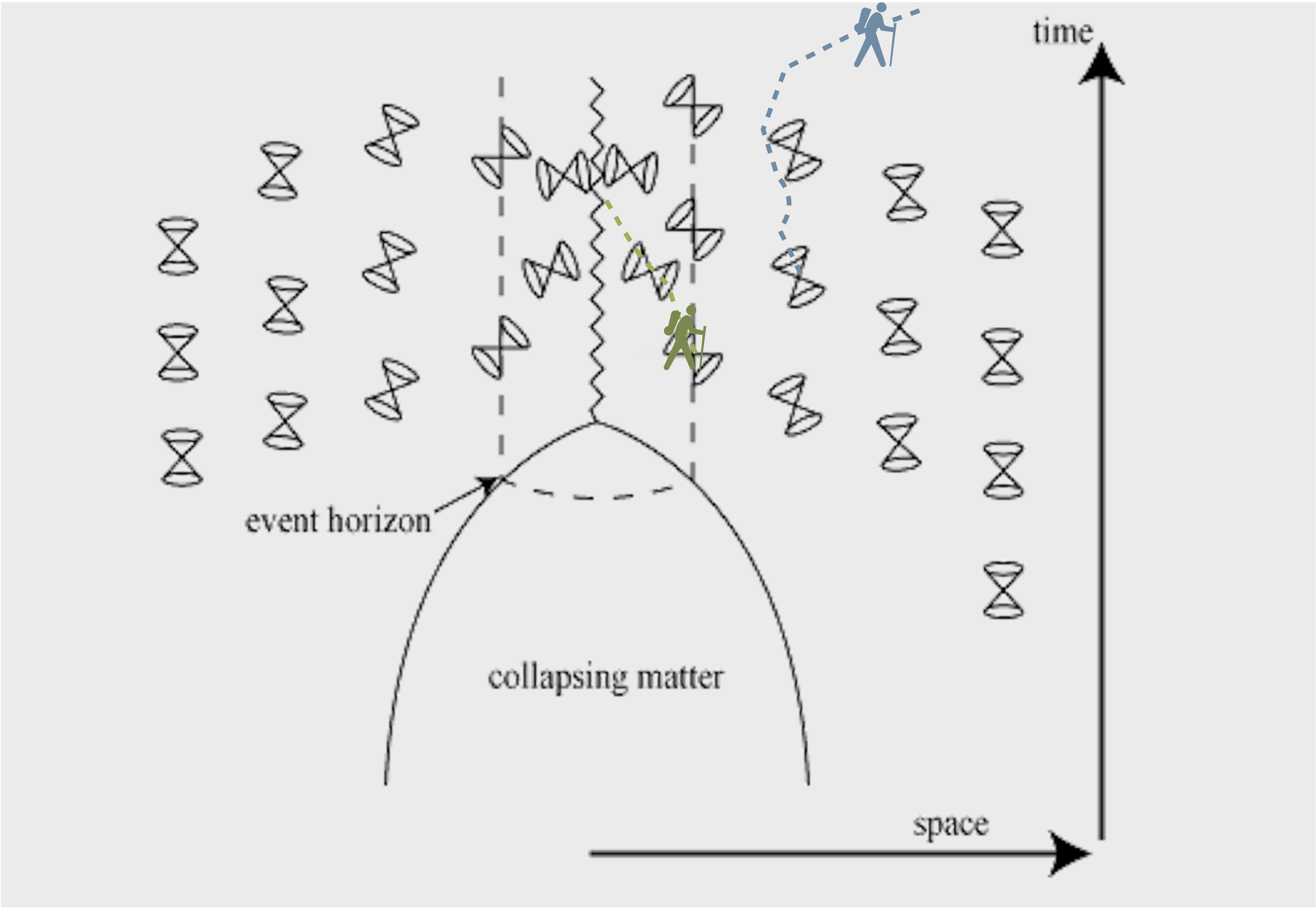
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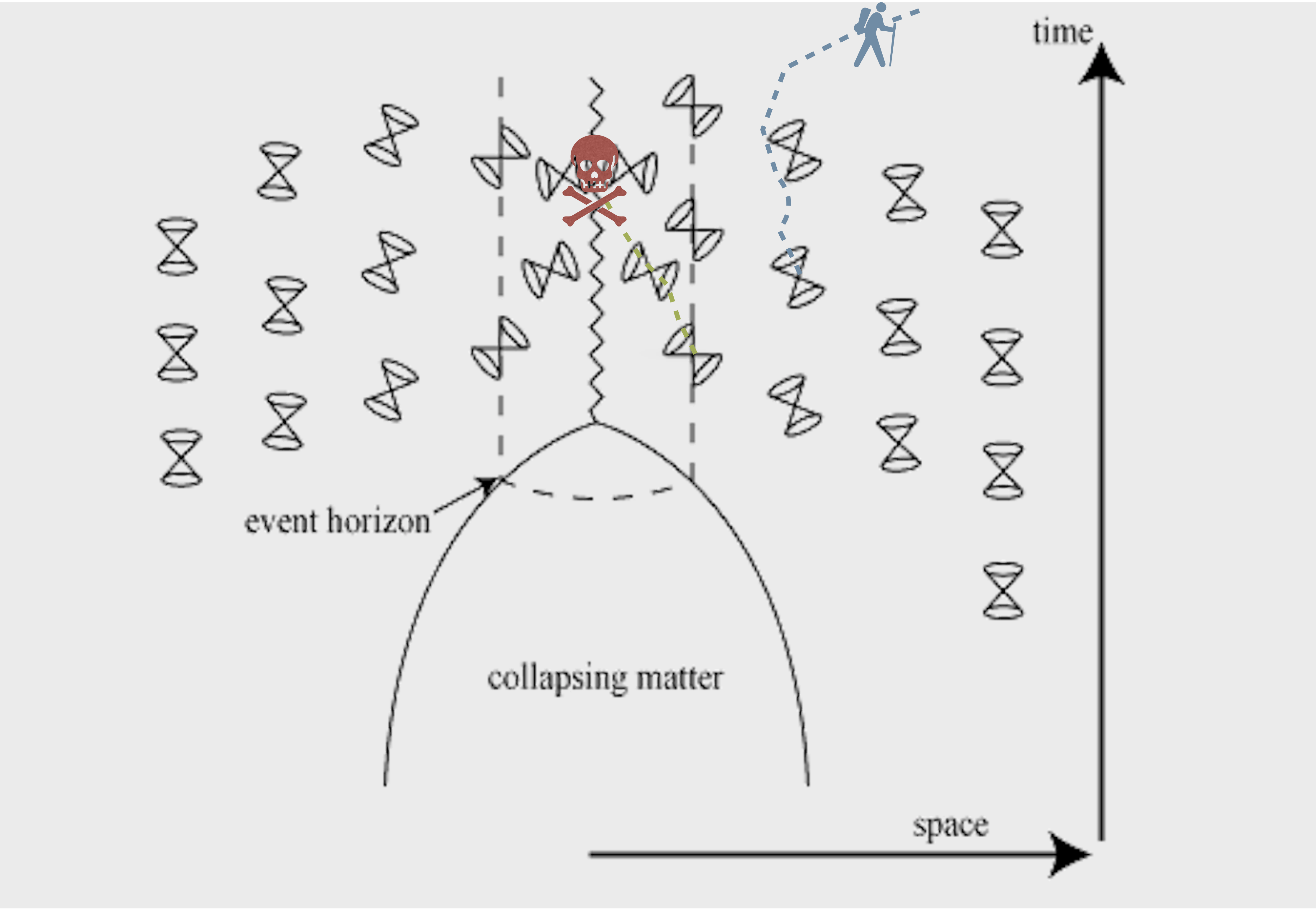
time
event horizon
collapsing matter
space

THE EINSTEIN EQUATION

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A spacetime is a 4 dimensional manifold M equipped with a Lorentzian metric g that satisfies the Einstein equation:

$$\mathrm{Ric}(g) - \frac{1}{2}R(g)g = T$$

where

- $\mathrm{Ric}(g)$ is the Ricci curvature of g,
- $R(g)$ is the scalar curvature of g,
- T is the stress-energy tensor of the matter fields present in the spacetime.

A **vacuum spacetime** is a spacetime satisfying the Einstein vacuum equation:

$$\mathrm{Ric}(g) = 0$$

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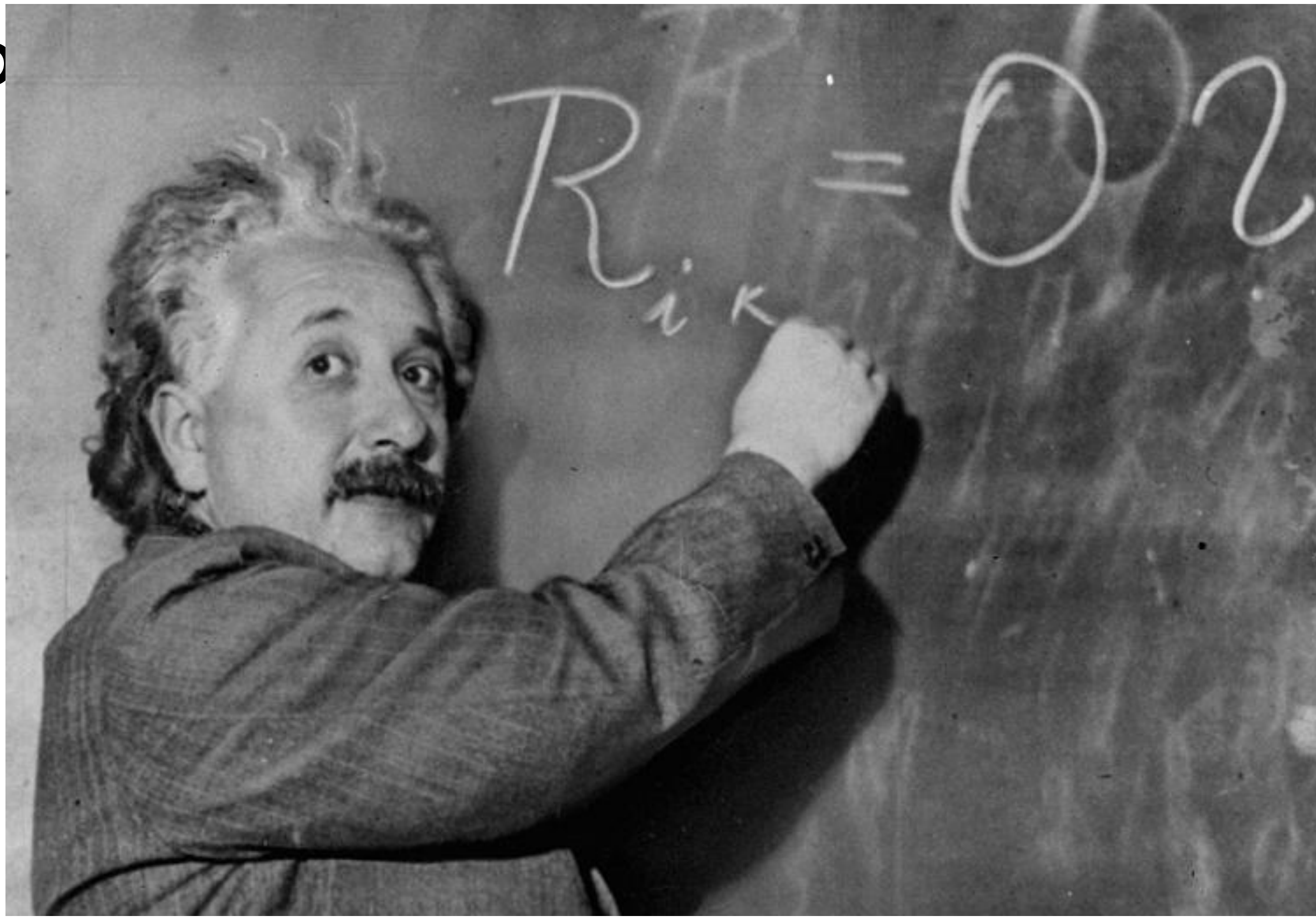
vacuum equation:

$$\mathrm{Ric}(g$$



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$$\mathrm{Ric}(g) = 0$$

An **electrovacuum spacetime** is a spacetime satisfying the Einstein-Maxwell equation:

$$\mathrm{Ric}(g) = 2F \cdot F - \frac{1}{2} |F|^2 g$$

where F is a 2-form, called the electromagnetic tensor, satisfying the Maxwell equations:

$$dF = 0, \qquad \operatorname{div} F = 0$$

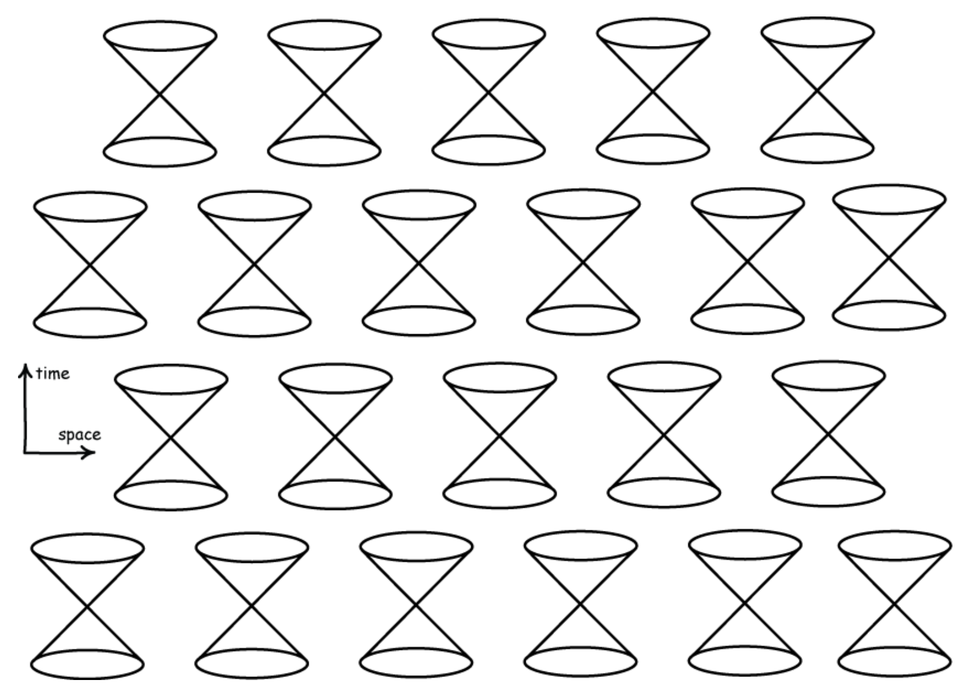
VACUUM SOLUTIONS

VACUUM SOLUTIONS

1. **Minkowski** spacetime (1905)

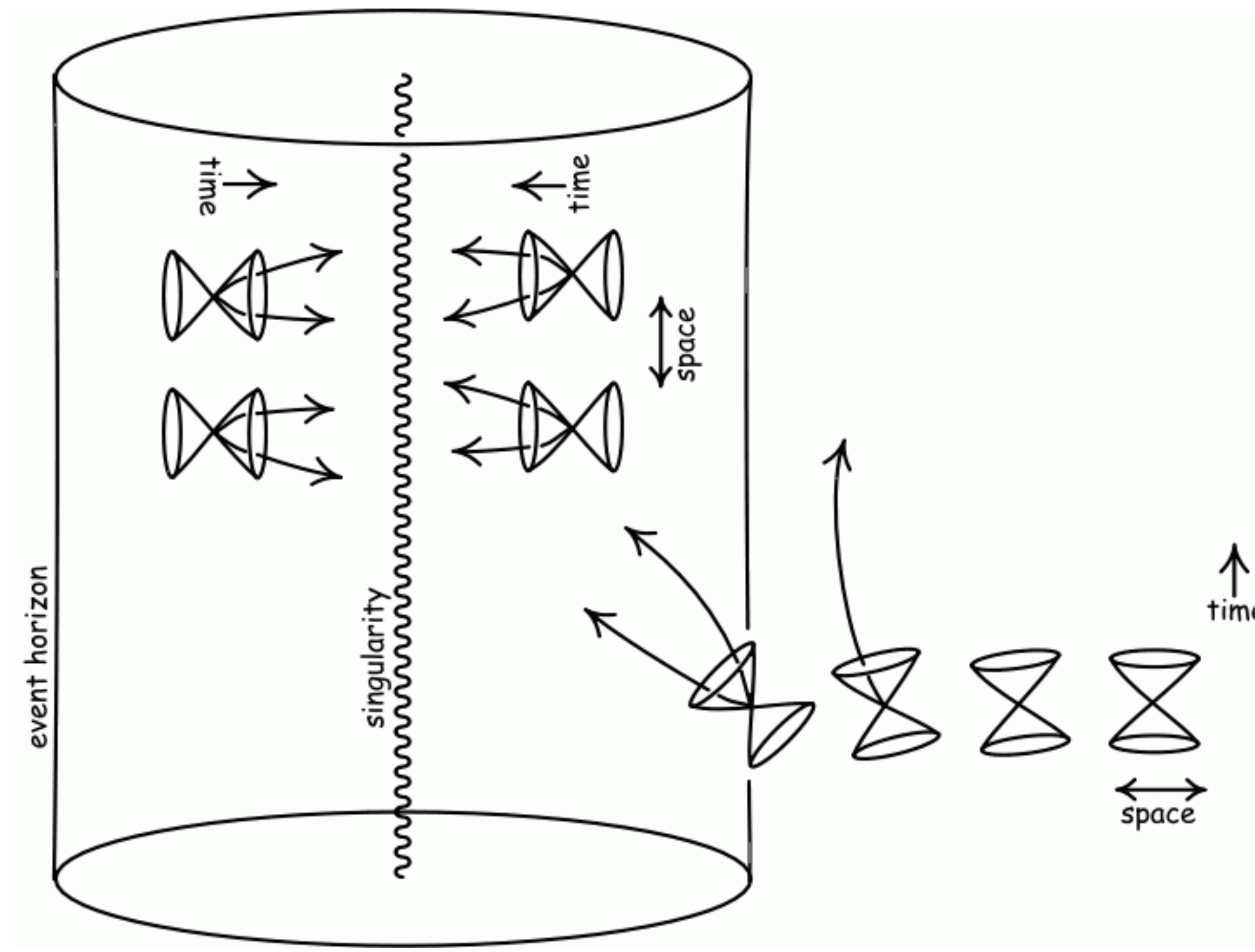
$$g_m = -\,dt^2 + dx^2 + dy^2 + dz^2$$

$$= -\,dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$



2. **Schwarzschild** spacetime (1916), for $M \in \mathbb{R}$

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$



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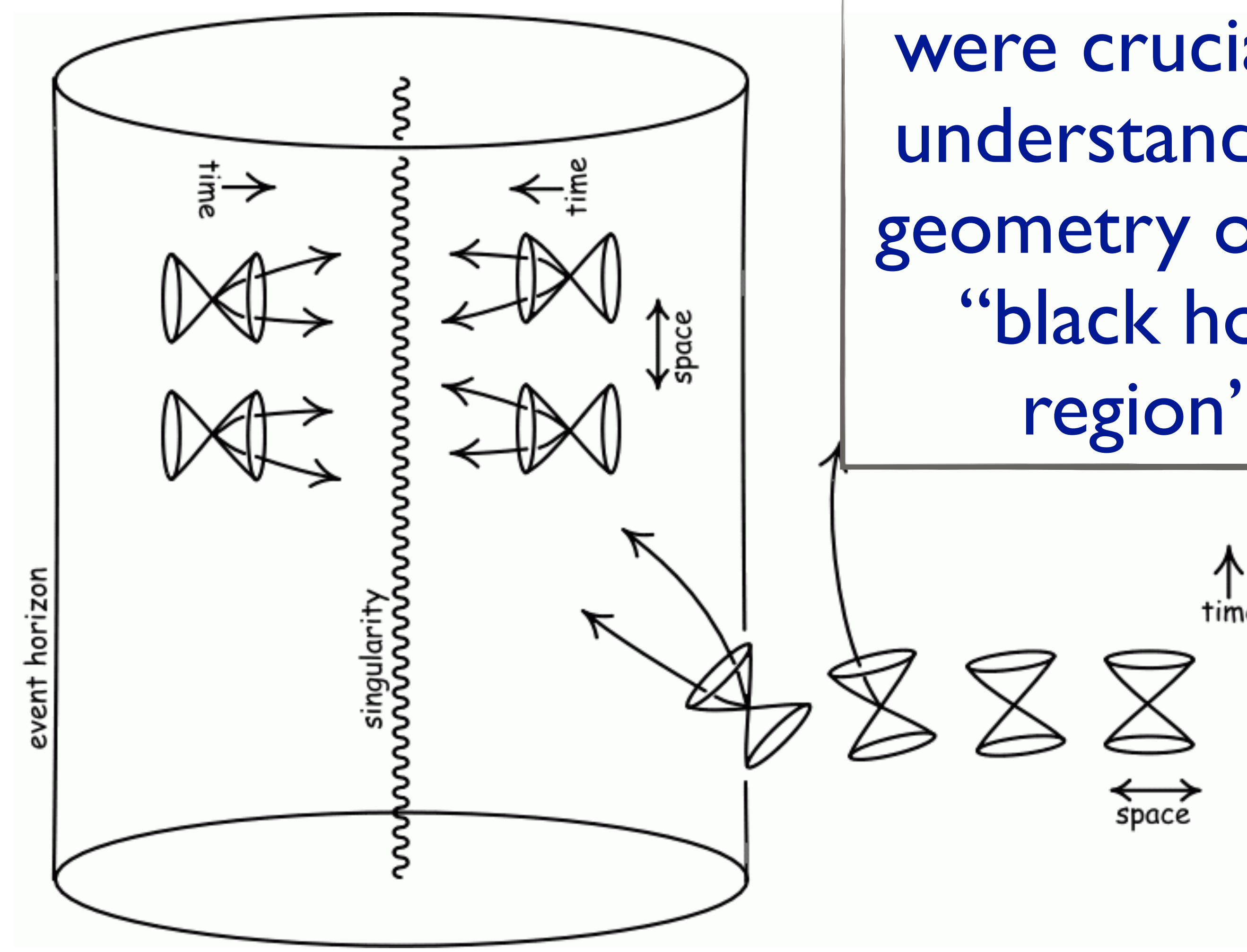
Mathematicians were crucial to understand the geometry of the "black hole region"



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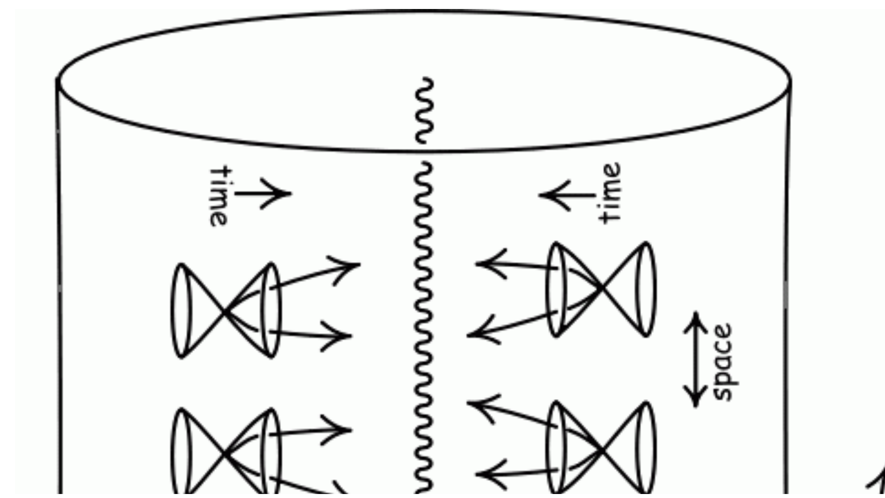

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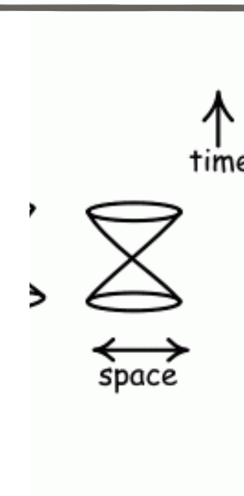

ANNALS OF MATHEMATICS
Vol. 40, No. 4, October, 1939

ON A STATIONARY SYSTEM WITH SPHERICAL SYMMETRY CONSISTING OF MANY GRAVITATING MASSES

BY ALBERT EINSTEIN

(Received May 10, 1939)

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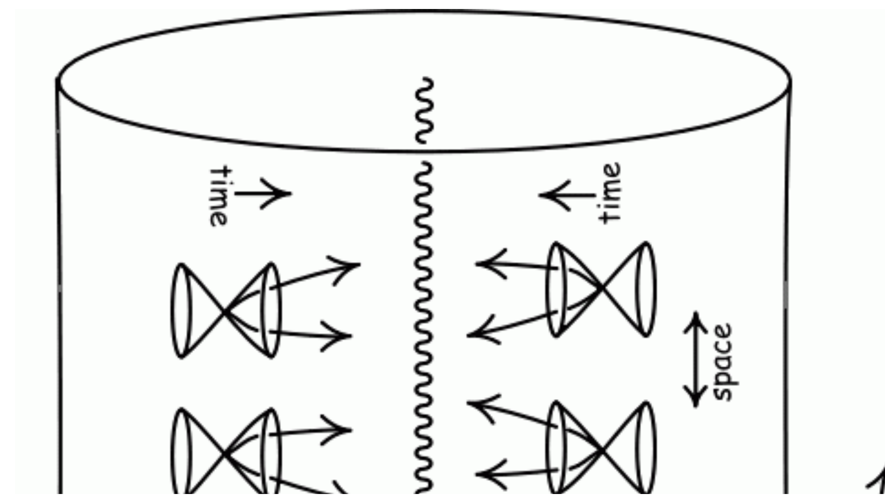

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ANNALS OF MATHEMATICS
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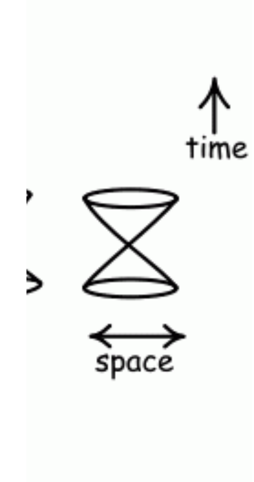

The essential result of this investigation is a clear understanding as to why the "Schwarzschild singularities" do not exist in physical reality. Although the

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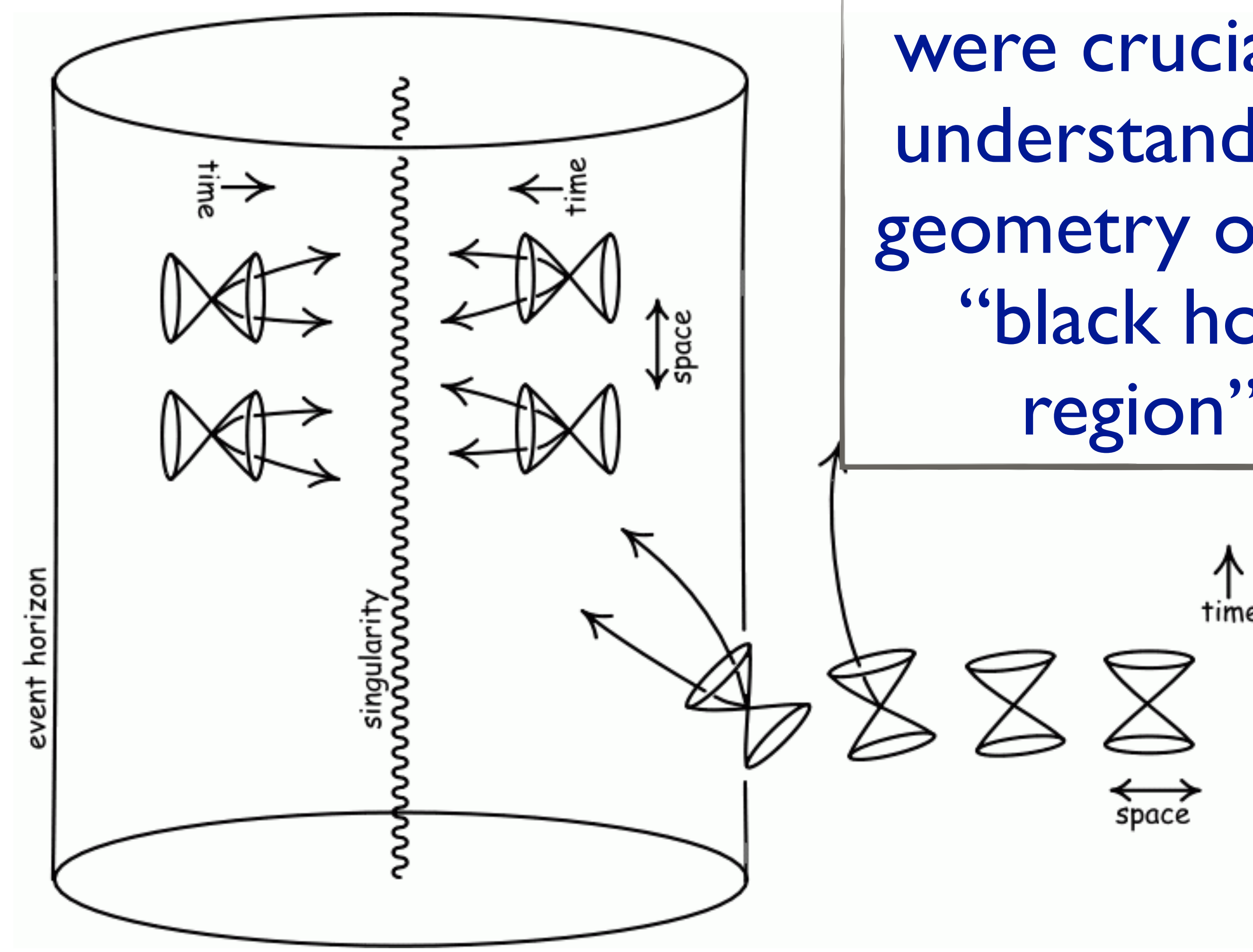
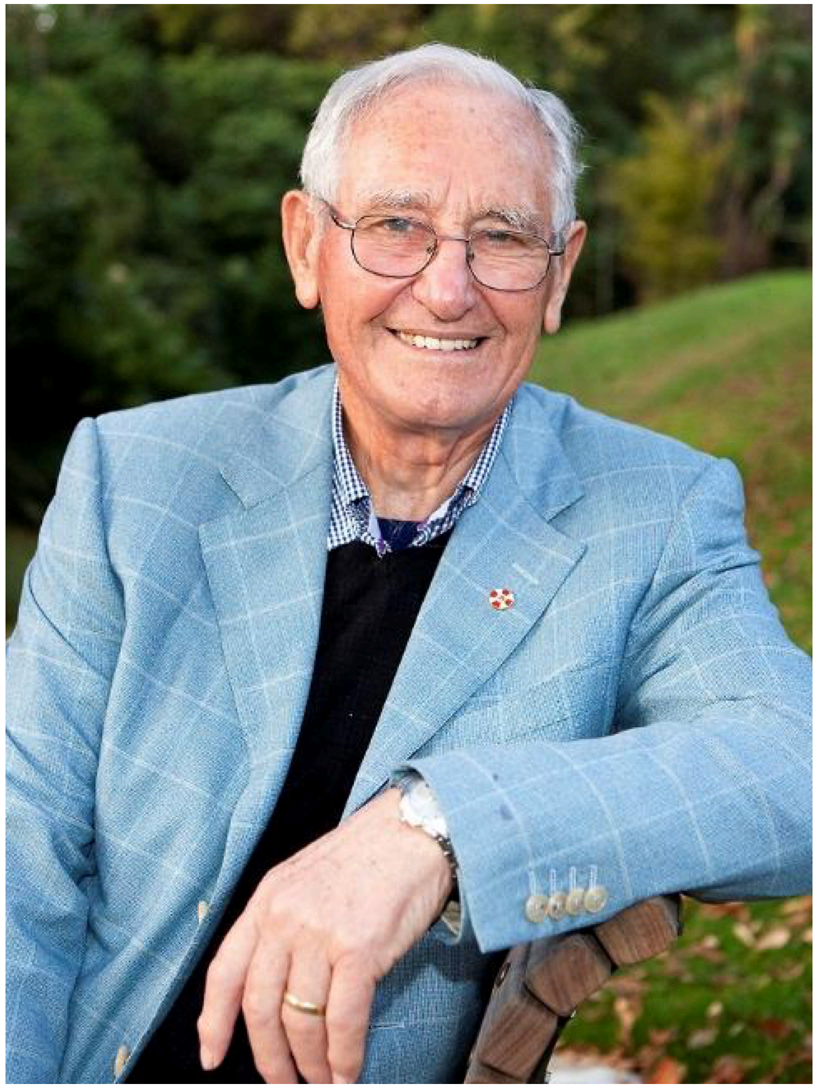


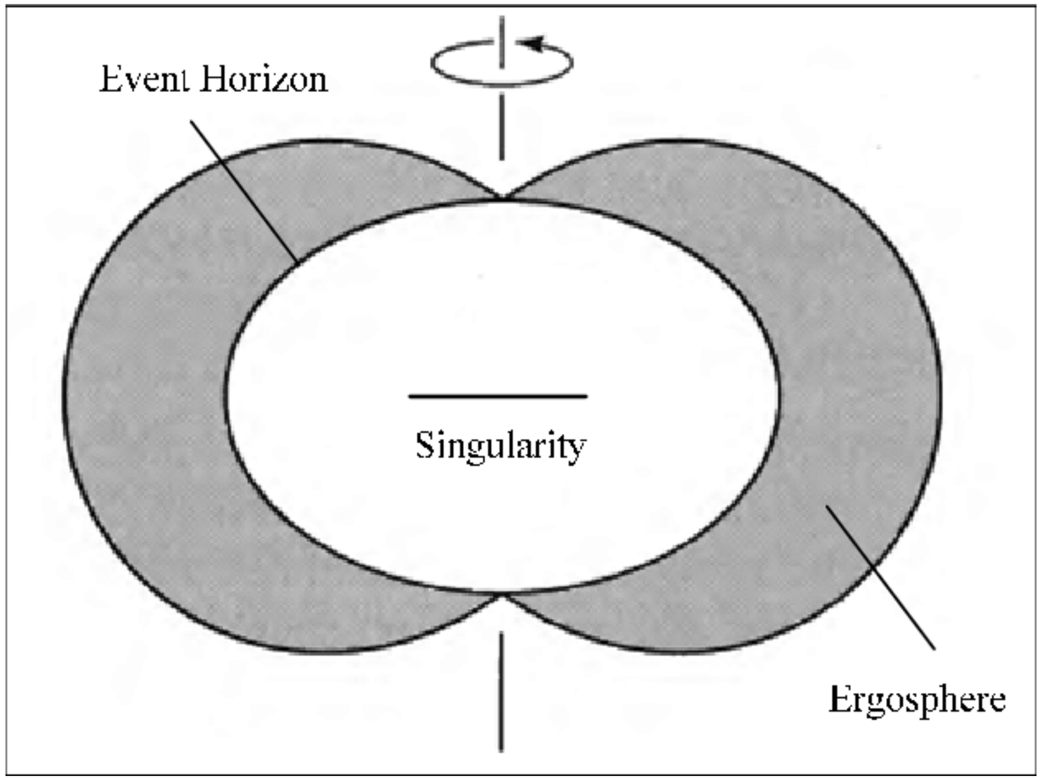
Mathematicians were crucial to understand the geometry of the "black hole region"

3. **Kerr** spacetime (1963), for $|a| \leq M$

$$g_{M,a} = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(adt - (r^2+a^2)d\phi\right)^2$$

where
$$\Delta = r^2 - 2Mr + a^2$$
$$\rho^2 = r^2 + a^2\cos^2\theta$$





ELECTROVACUUM SOLUTIONS

ELECTROVACUUM SOLUTIONS

1. **Reissner-Nordström** spacetime (1917), for $|Q| \leq M$

$$g_{M,Q} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

ELECTROVACUUM SOLUTIONS

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2. **Kerr-Newman** spacetime (1965), for $\sqrt{a^2 + Q^2} \leq M$

$$g_{M,a,Q} = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left(a dt - (r^2 + a^2) d\phi\right)^2$$

where

$$\Delta = r^2 - 2Mr + a^2 + Q^2$$
$$\rho^2 = r^2 + a^2\cos^2\theta$$

THE DYNAMICS OF BLACK HOLES

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Theorem [Choquet-Bruhat(1952)]

The Einstein equation in wave coordinates is given by

$$\Box_g g = \mathcal{N}(g, \partial g)$$

with initial data $(g|_{\Sigma_0}, k|_{\Sigma_0})$, where $\Box_g = g^{\alpha\beta}\nabla_\alpha\nabla_\beta$ is the D'Alembertian operator.

THE DYNAMICS OF BLACK HOLES

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In Minkowski,

$$\Box_{g_m} = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$$

THE DYNAMICS OF BLACK HOLES

Theorem [Choquet-Bruhat(1952)]

The Einstein equation in wave coordinates is given by

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with initial data $(g|_{\Sigma_0}, k|_{\Sigma_0})$,

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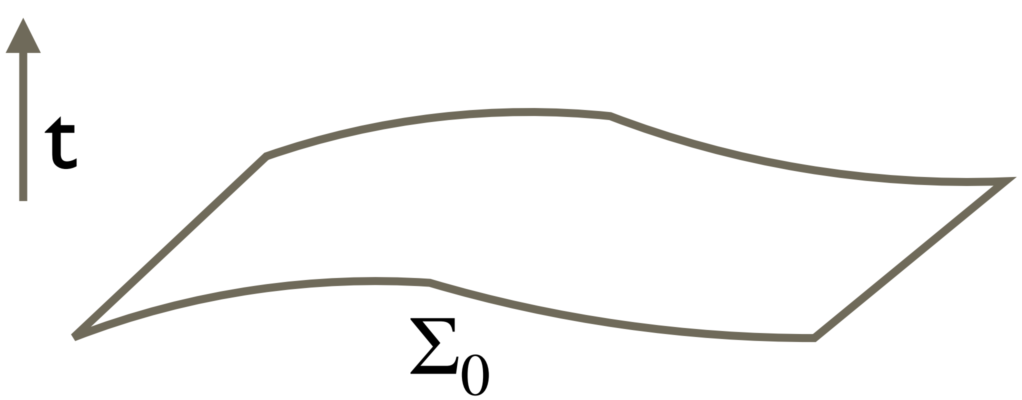
$$\Box_{g_m} = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$$



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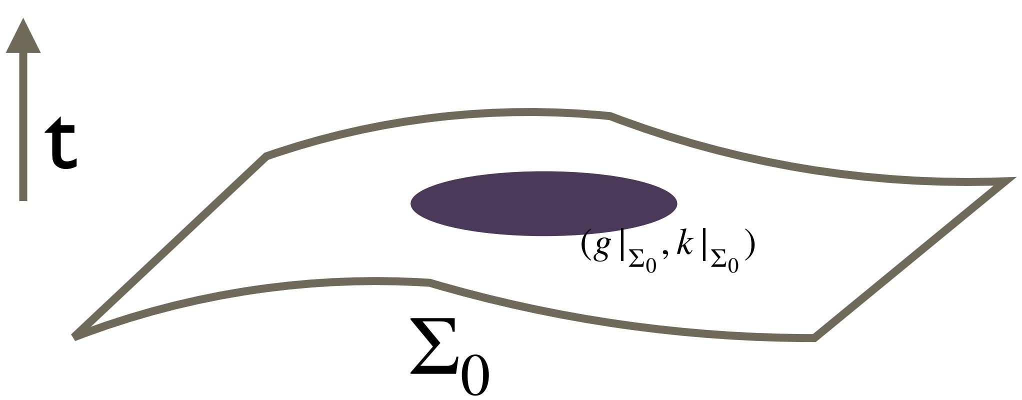

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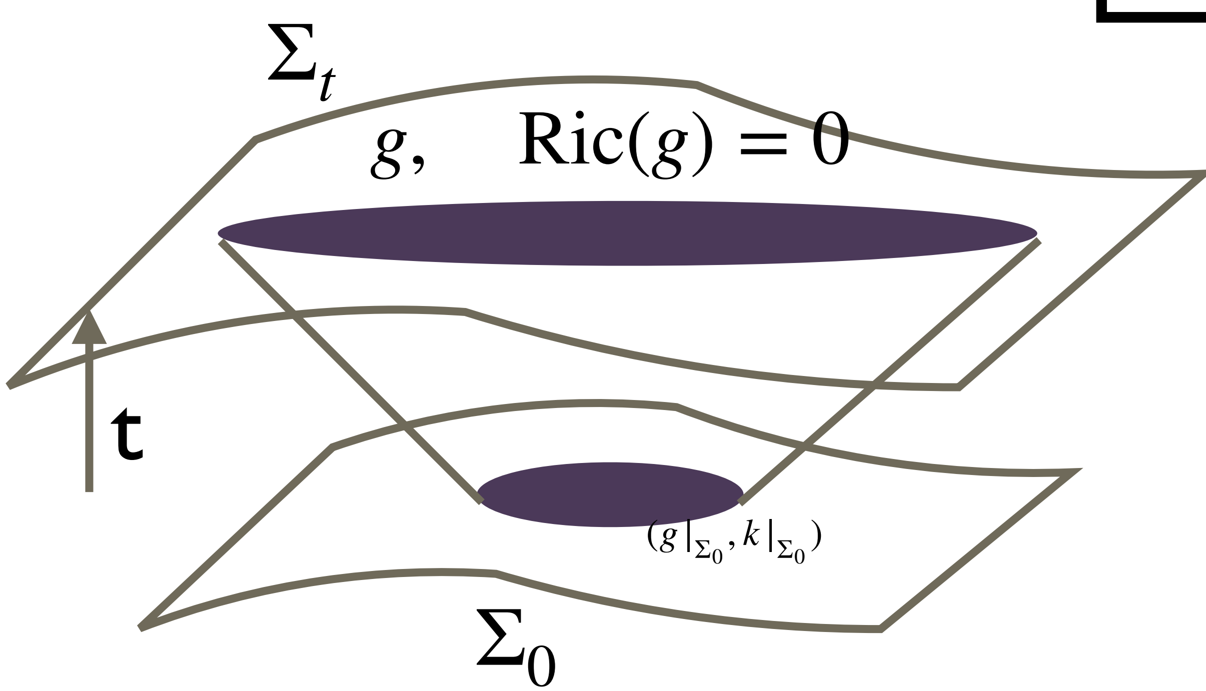

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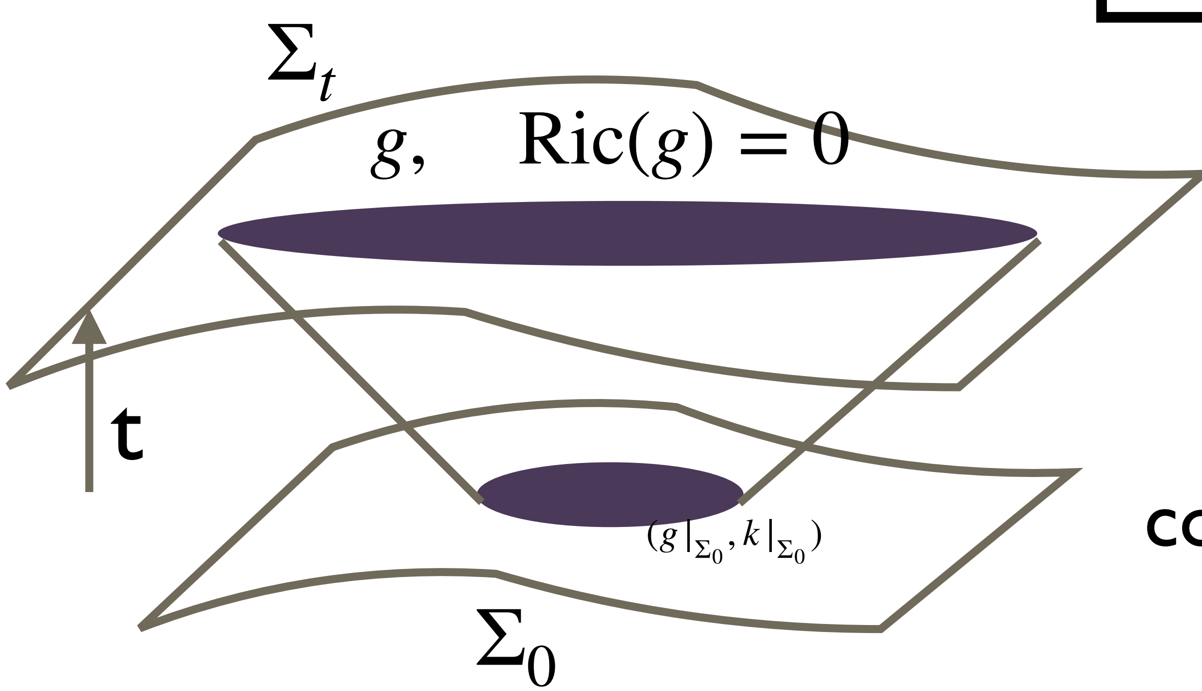

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This implies local well-posedness and continuous dependence on the initial data.

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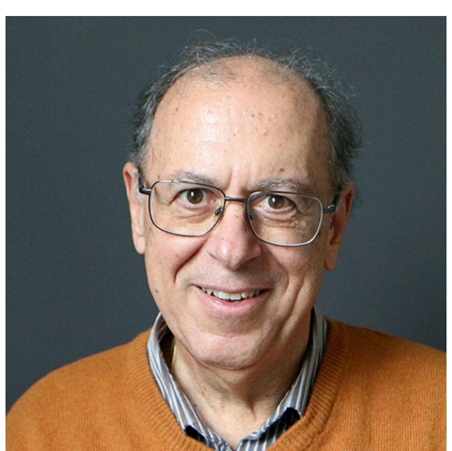
It turns out that this does not happen!



Theorem
[Christodolou-Klainerman(1993)]

Minkowski is globally non-linearly stable.

What about the global behavior of perturbations of non-trivial solutions to the Einstein equation, like black holes?

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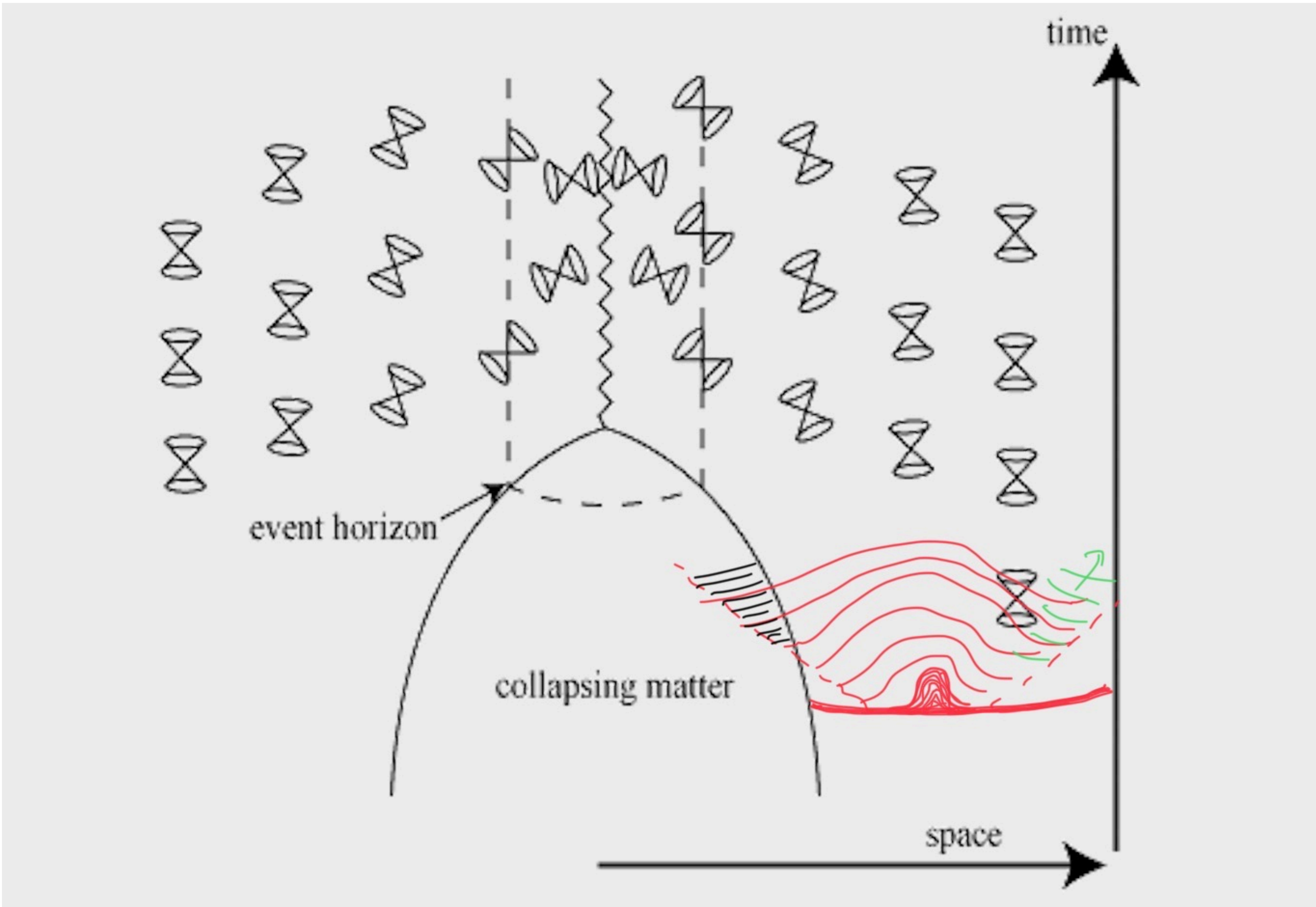

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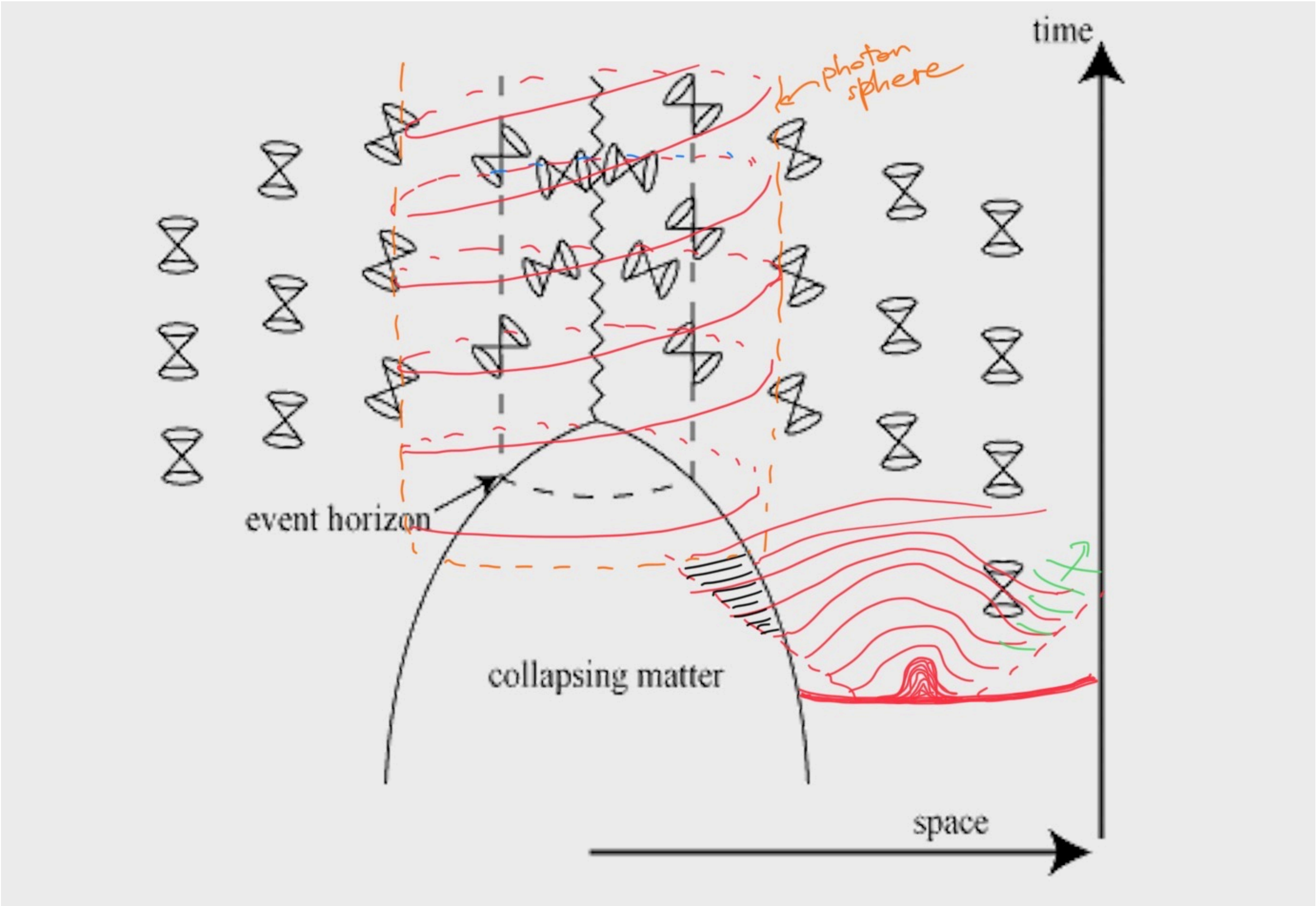
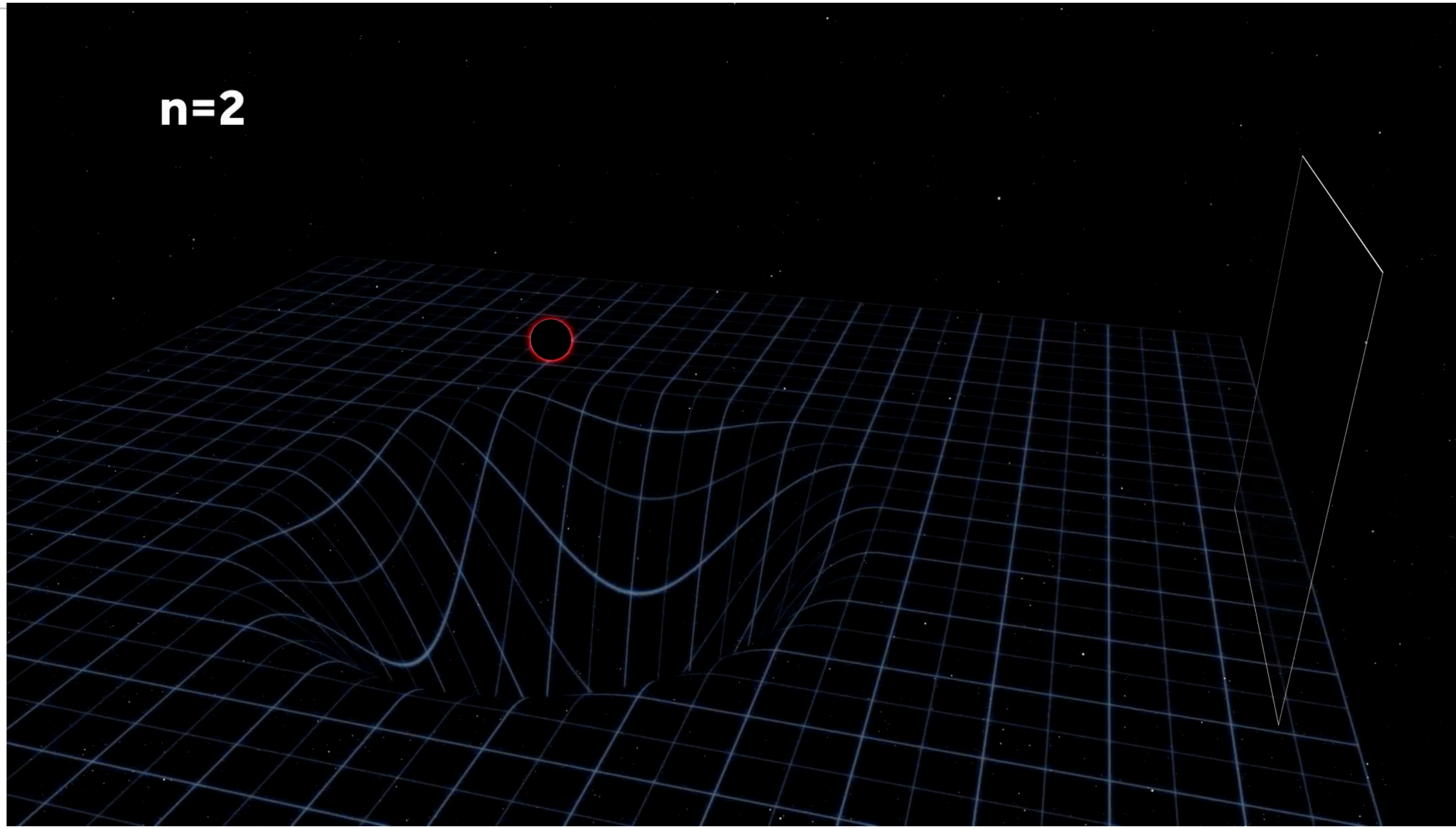
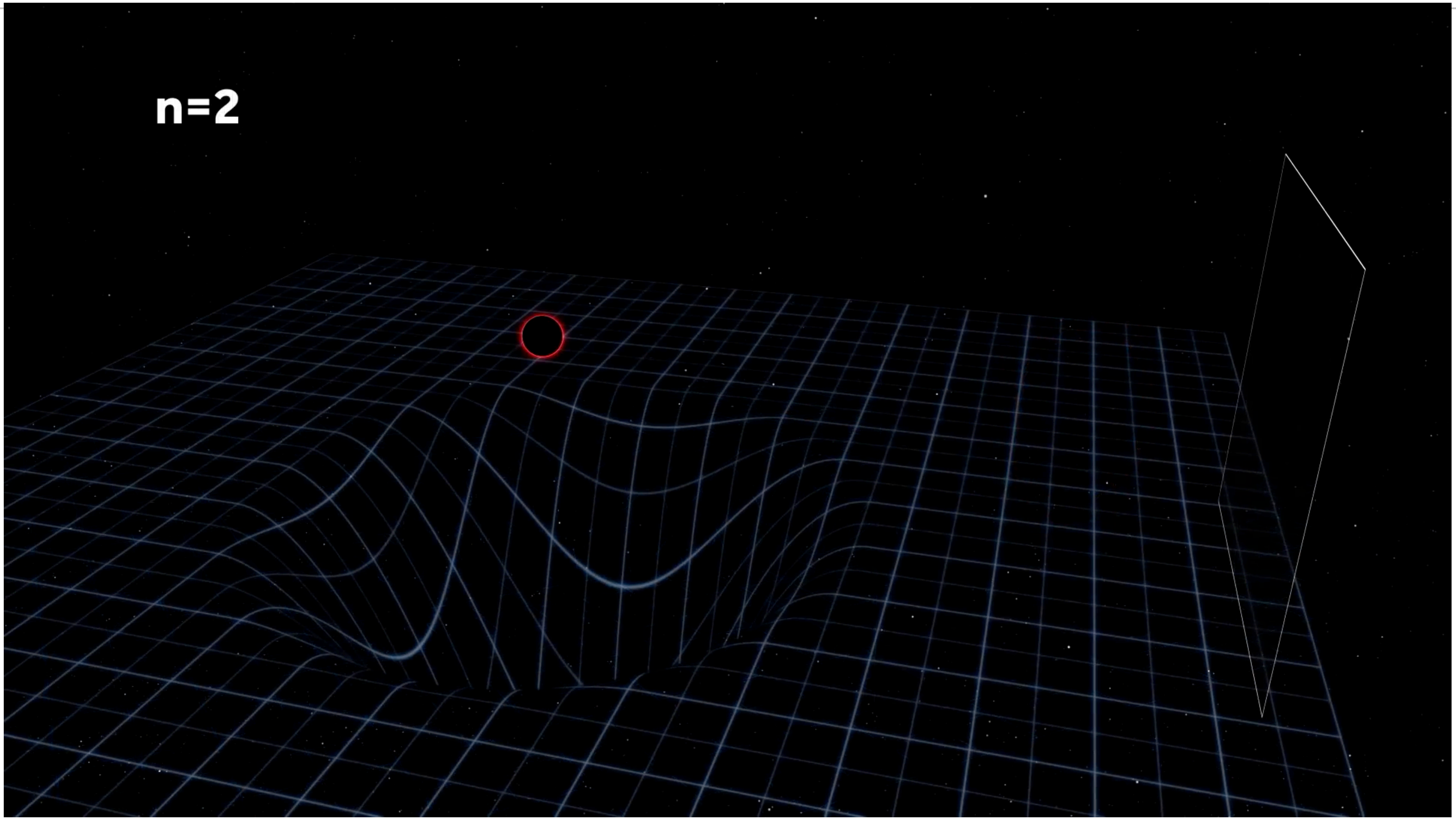


Outside a black hole, there is a region of trapped null geodesics


n=2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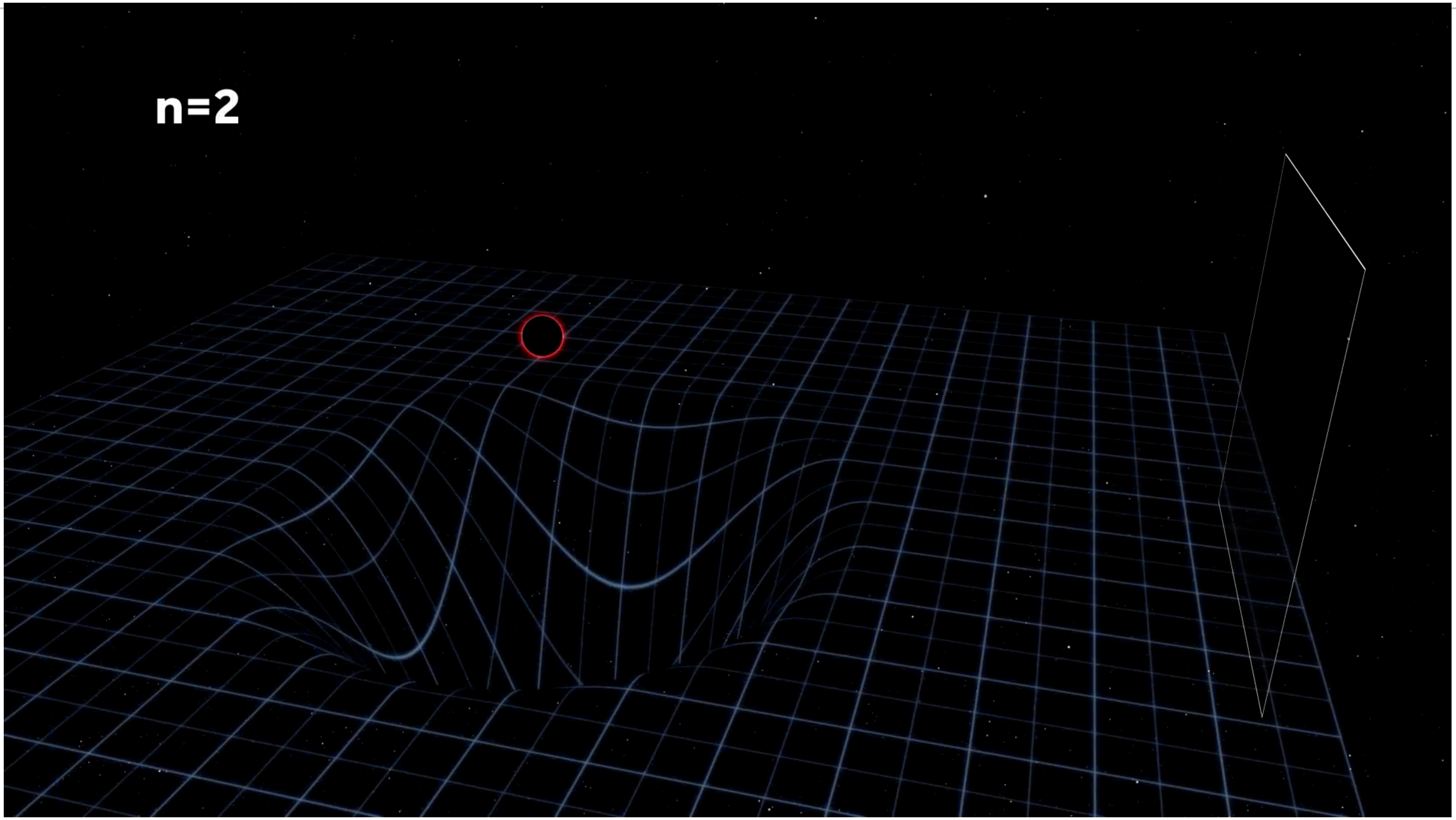
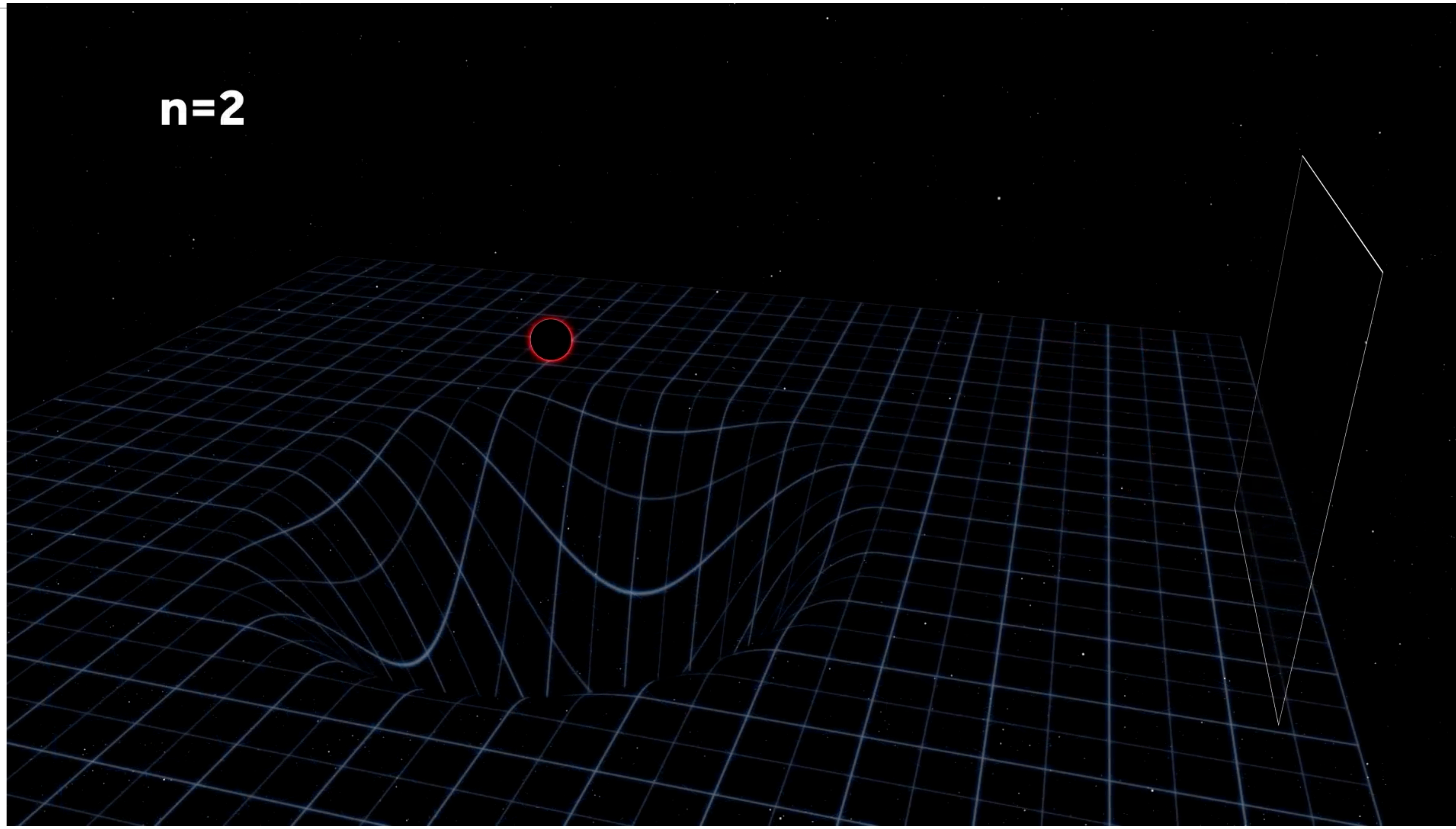


Science Alert, Event Horizon Telescope

The trapped null geodesics are unstable, so they tend to scatter off.



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Theorem
[Dafermos-Rodnianski-Shlapentokh-Rothman (2014)]

The wave equation $\Box_{g_{M,a}} \phi = 0$ on rotating black holes is stable.

OUTLINE

1. Motivation for the study of black holes
2. The mathematics of black holes
3. The stability problem for the Einstein equation
4. Black holes with matter fields

THE FINAL STATE CONJECTURE

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Initial data for the Einstein equation evolve asymptotically in time to a finite number of Kerr-Newman black holes, moving away from each other.

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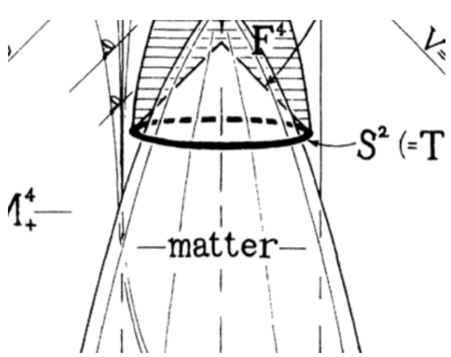

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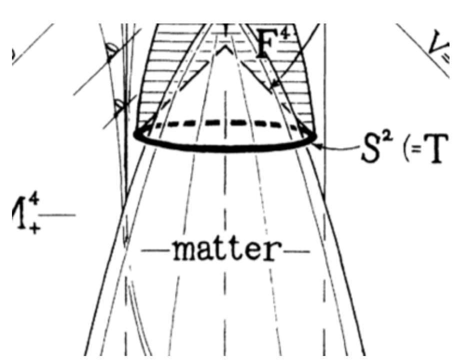

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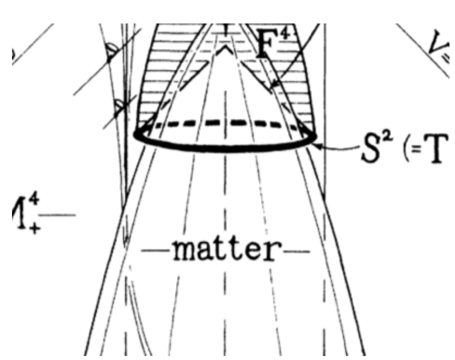

RIGIDITY CONJ:

Kerr-Newman is the only family of stationary solutions to the Einstein equation ("no-hair theorem").

STABILITY CONJ:

The Kerr-Newman family is stable under small perturbations of the initial data.

STABILITY OF KERR-NEWMAN CONJECTURE

Initial data for the Einstein equation which are sufficiently close to a Kerr-Newman black hole evolve asymptotically in time to another member of the Kerr-Newman family.

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Translation in PDE language:

Schematically, the Einstein equation is a non-linear PDE

$$\mathcal{N}[\phi] = 0 \qquad (1)$$

with a family of stationary solutions ϕ_λ, i.e. $\mathcal{N}[\phi_\lambda] = 0$.

We want to show that every solution ϕ of (1) with initial data close to a ϕ_λ converges asymptotically in time to a $\phi_{\lambda'}$ for some λ' close to λ.

One first looks at the linearized equation around a solution ϕ_λ

$$(d\mathcal{N})\big|_{\phi_\lambda}(\psi) = 0 \qquad (2)$$

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There are two levels of increasing difficulty:

1. One can only look at special *mode solutions*, of the separated form

$$\psi(t, r, \theta, \varphi) = e^{-i\omega t} e^{im\varphi} S(\theta) R(r)$$

and show that there are no exponentially growing modes: **mode stability**

2. One can prove that general solutions of (2) are bounded and decay in time: **full linear stability**

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In August 2017, LIGO observed the first merger of two neutron stars. Just two seconds after the gravitational wave signal was detected, a flash of gamma-ray was detected by the FERMI satellite, coming from the same tiny corner of the cosmos.

NASA's Goddard Space Flight Center/CI Lab

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The merger of two neutron stars is expected to relax to a charged Kerr-Newman black hole

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2. One cannot expect to have just one wave equation to analyze, but rather a system of coupled wave equations which describes the interaction between the gravitational and the electromagnetic radiation:

$$\underbrace{\mathrm{Ric}(g)}_{\substack{\text{gravitational radiation} \\ \text{(2-tensor)}}} = \underbrace{2F \cdot F - \frac{1}{2} g \, |F|^2}_{\substack{\text{electromagnetic radiation} \\ \text{(1-tensor)}}}$$

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(3) Ann. Henri Poincaré, 21, 2485 - 2580 (2020), (4) Comm. Math. Phys. (2020), (5) Annals of PDE, 6, 8 (2020)

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- Identification of gauge invariant quantities $\mathfrak{b}$ and $\mathfrak{f}$ representing gravitational and electromagnetic radiation, which satisfy a system of coupled Teukolsky equations[(1)(2)(3)]

$$\mathscr{T}(\mathfrak{f}) = Q \cdot \nabla(\mathfrak{b})$$
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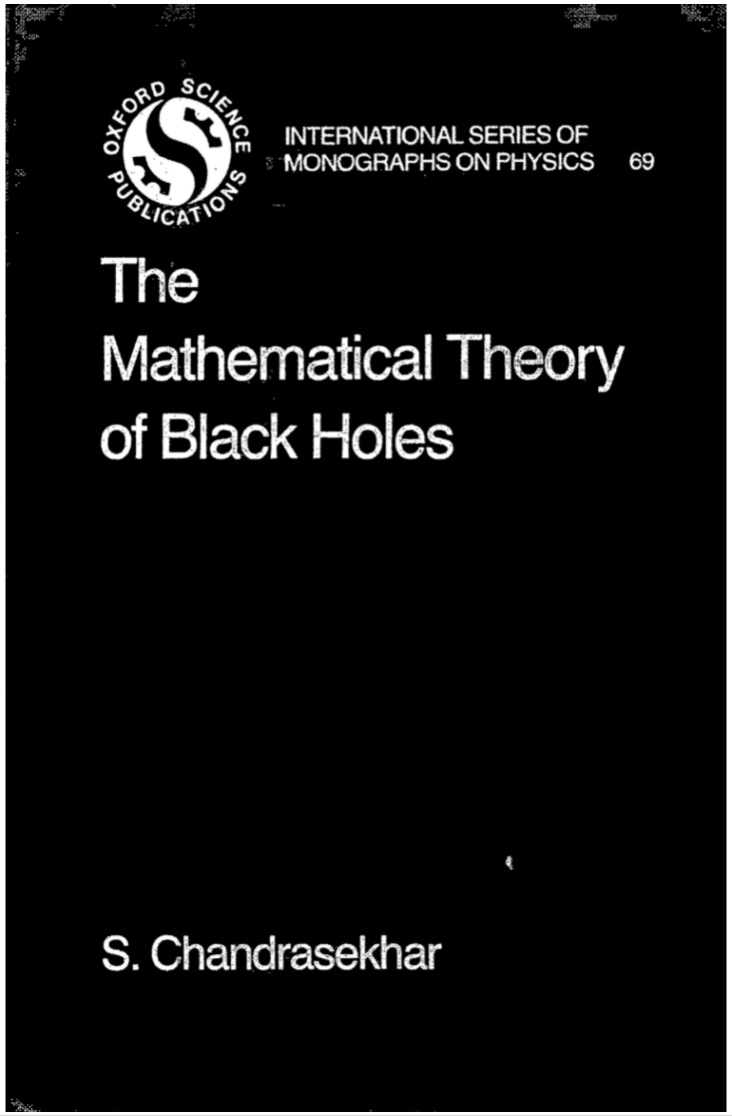

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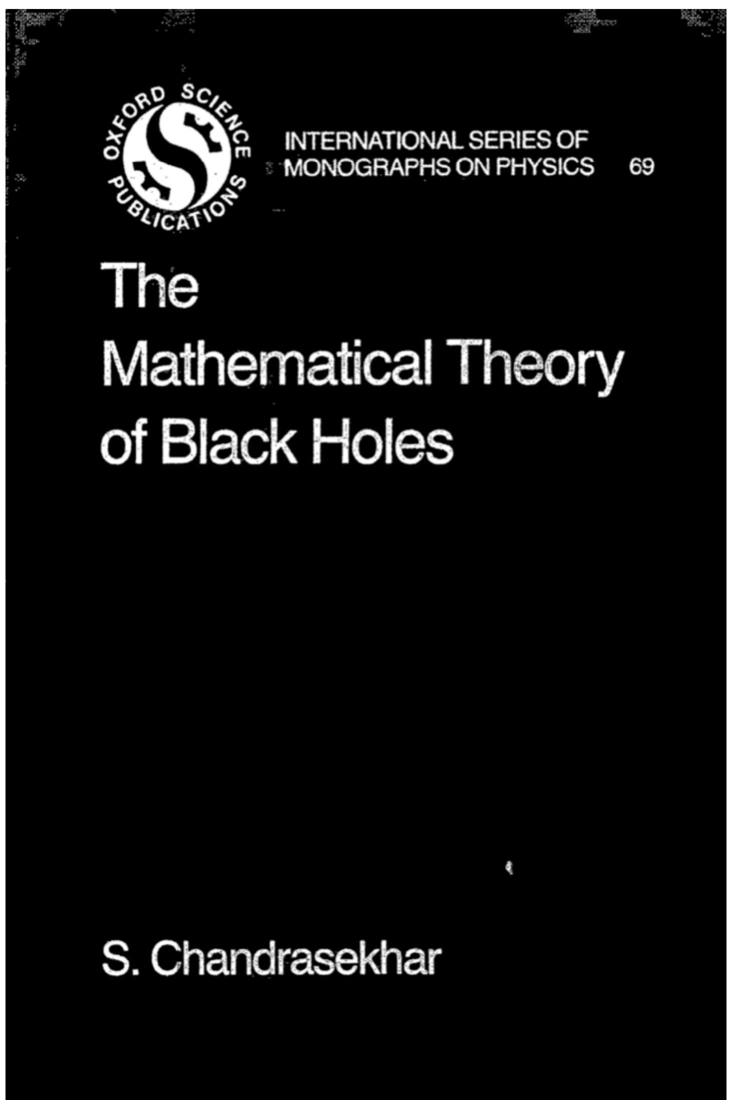

111. The equations governing the coupled electromagnetic-gravitational perturbations of the Kerr–Newman space-time

As we have stated in the introductory section (§108), the methods that have proved to be so successful in treating the gravitational perturbations of the

Kerr space-time do not seem to be applicable (nor susceptible to easy generalizations) for treating the coupled electromagnetic-gravitational perturbations of the Kerr–Newman space-time. The principal obstacle is in finding separated equations. In this section, we shall briefly consider the origin of this apparent indissolubility of the coupling between the spin-1 and spin-2 fields in the perturbed space-time.

The issue relies in the decomposition in modes in an axially symmetric (and not spherically symmetric) background.

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and all efforts to decouple (or separate) these equations were not successful. And numerous other alternative manipulations of the system of equations (139)–(142) were equally unsuccessful.

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If one can prove boundedness of a general solution through a physical space analysis, it will then in particular imply the absence of exponentially growing modes, therefore proving mode stability!

$\boxed{\text{full linear stability}} \longrightarrow \boxed{\text{mode stability}}$

THE PHYSICAL SPACE EQUATIONS IN KERR-NEWMAN

(1) arXiv:2002.07228

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It makes a mathematician very happy to know that her rigorous proof of stability of the charged rotating black hole is the way to succeed where the mode analysis in physics failed.

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CONCLUSIONS
[PHILOSOPHICAL NOTE]

Physics challenges us with interesting mathematical problems, and mathematicians' contribution is often in the rigorous analysis of the objects and concepts already understood in a heuristic way.

But actually, just like in the case of black holes, mathematics can be crucial in shedding light on physical phenomena which would not be understood otherwise.

THANK YOU
FOR YOUR ATTENTION!